SOLIDS MECHANICS

UNIT I: STRESSES AND STRAINS

Syllabus:

UNIT – I: Simple Stresses & Strains : Elasticity and plasticity – Types of stresses & strains– Hooke's law – stress – strain diagram for mild steel – Working stress – Factor of safety – Lateral strain, Poisson's ratio & volumetric strain – Bars of varying section – composite bars – Temperature stresses- Relation between elastic constants- principal stresses-Mhor's circle

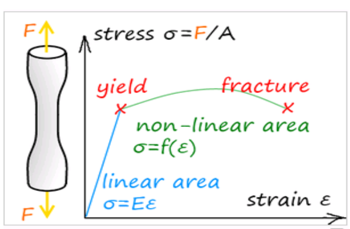
INTRODUCTION

Strength of materials/Mechanics of solids deals with the relations between the external forces applied to elastic bodies with the resulting deformations and stresses. In the design of structures and machines, the application of the principles of strength of materials is necessary if satisfactory materials are to be utilized and adequate proportions obtained to resist functional forces.

Forces are produced by the action of "gravity, by accelerations and impacts of moving parts, by gasses and fluids under pressure, by the transmission of mechanical power", etc. In order to analyze the stresses and deflections of a body, the magnitudes, directions and points of application of forces acting on the body must be known. Information given in the Mechanics section provides the basis for evaluating force systems.

Mechanical properties

Strength: Ability of a material to resist the externally applied load without failure

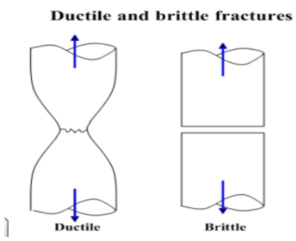


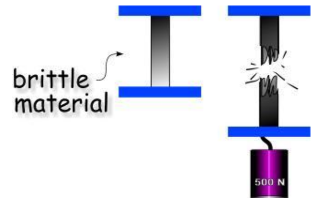
Basic static response of a specimen under tension

Brittleness:

Tendency of a material to fracture or fail upon the application of a relatively small amount of force, impact or shock without much plastic deformation.

Eg: Cast iron, glass, ceramics





Creep:

When a metal is subjected to a constant force at high temperature below its yield point, for a prolonged period of time, it undergoes a permanent deformation called creep.

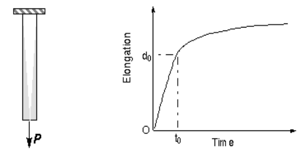
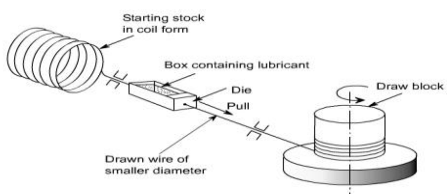


Figure 11. Creep in a bar under constant load.

Eg: gas turbine blade

Ductility:

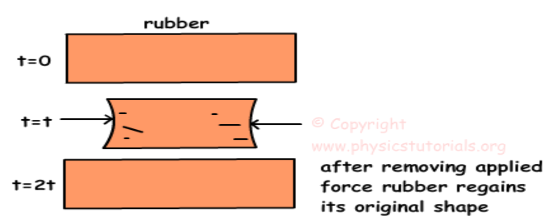
Ductility is the property by which a metal can be drawn into thin wires. It is determined by percentage elongation and percentage reduction in area of a metal.



Eg: steel, copper, aluminum

Elasticity:

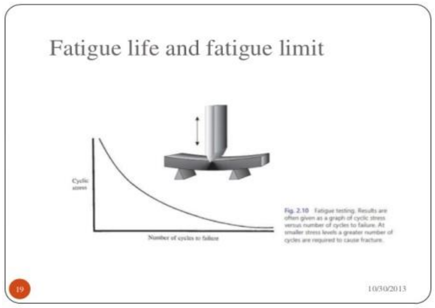
Elasticity is the tendency of solid materials to return to their original shape after being deformed.



Eg: Rubber, steel

Fatigue:

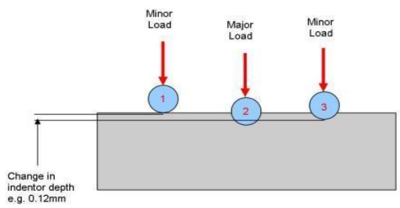
Fatigue is the material weakening or breakdown of material subjected to stress, especially a repeated series of stresses.



Eg: Suspension of an automobile

Hardness:

Hardness is the ability of a material to resist scratches and indentation also resist permanent change of shape caused by an external force.



Eg:Diamond, Cast iron

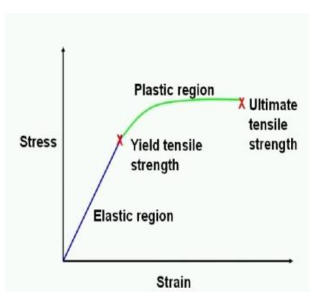
Malleability:

Malleability is the property by which a metal can be rolled into thin sheets.

Eg: steel sheets, Al sheets

Plasticity:

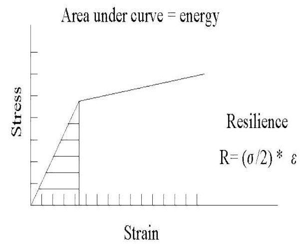
Plasticity is the property by which a metal retains its deformation permanently, when the external force applied on it is released.



Eg: Deep drawing of sheet metals, clay, lead

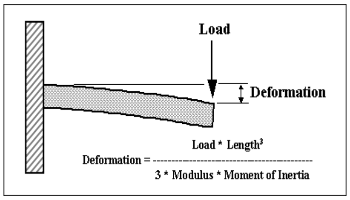
Resilience:

Resilience is the ability of a metal to absorb energy and resist soft and impact load or energy stored in a material within elastic limit.



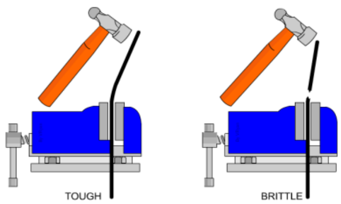
Stiffness:

When an external force is applied on a metal, it develops an internal resistance. The internal resistance developed per unit area is called stress. Stiffness is the ability of a metal to resist deformation under stress.

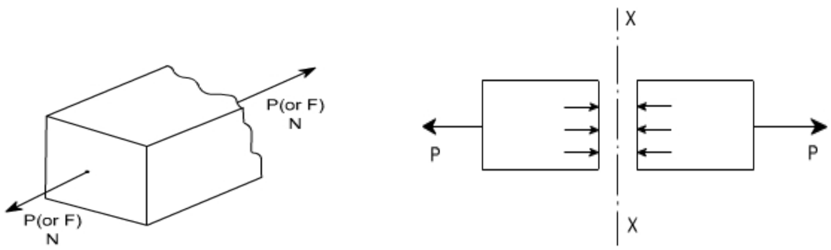


Toughness:

When a huge external force is applied on a metal, the metal will experience fracture. Toughness is the ability of a metal to resist fracture or store energy. The area under stress strain diagram represents toughness only



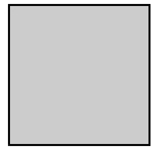
STRESS



Stress is defined as the force intensity or resisting force per unit area. Here we use a symbol to represent the stress by sigma.

Therefore, $\sigma = P/A$

Where A is the area of the X – section



The basic units of stress in S.I units i.e. (International system) are

As *Pascal* is a small quantity, in practice, multiples of this unit is used.

1 kPa = 10^3 Pa = 10^3 N/ m^2 (kPa = Kilo Pascal)

1 MPa = 10^6 Pa = 10^6 N/ m^2 = 1 N/mm^2 (MPa = Mega Pascal)

1 GPa = 10^9 Pa = 10^9 N/ m^2 (GPa = Giga Pascal)

TYPES OF STRESSES:

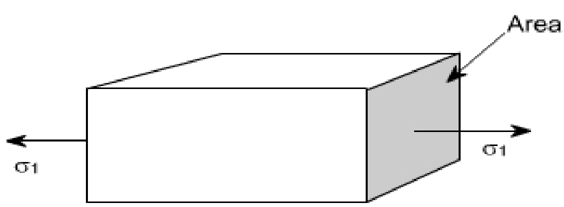
Only two basic stresses exists:

(1) Normal stress and

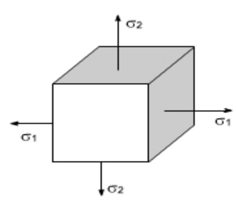
(2) Shear stress.

Other stresses either are similar to these basic stresses or a combination of this e.g. bending stress is a combination of tensile, compressive and shear stresses. Torsion stress, as encountered in twisting of a shaft is a shearing stress.

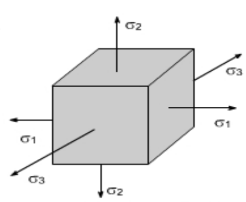
Normal stresses: We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses.



(uni-axial state of stress)



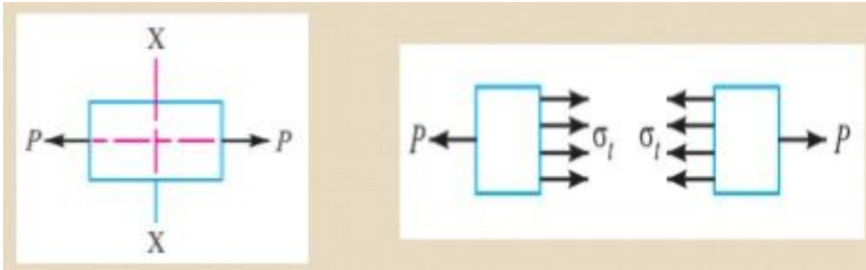
(Biaxial state of stress)

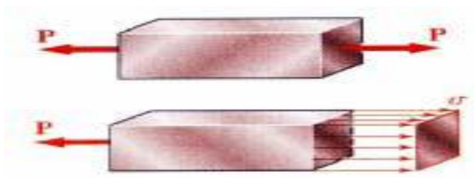


(Triaxial state of stress)

Tensile stress

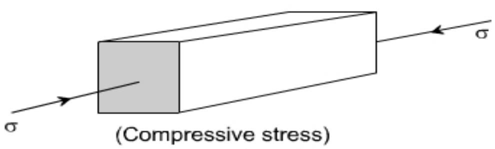
When a body is subjected to two equal and opposite axial pulls (also known as tensile load), then the stress induced at any section of the body is known as tensile stress

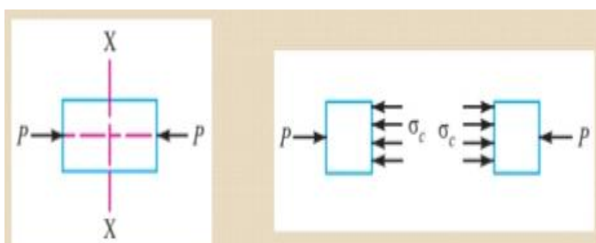




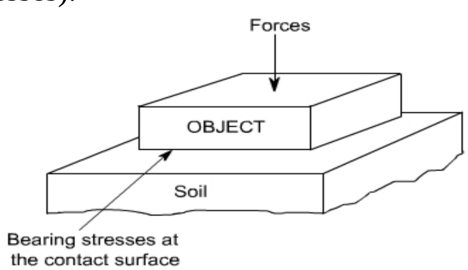
Compressive stresses

When a body is subjected to two equal and opposite axial pushes (also known as compressive load), then the stress induced at any section of the body is known as compressive stress.





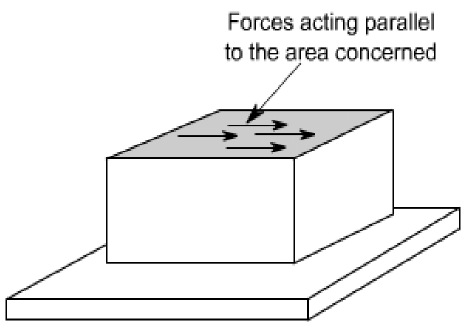
Bearing Stress: When one object presses against another, it is referred to a bearing Stress (They are in fact the compressive stresses).

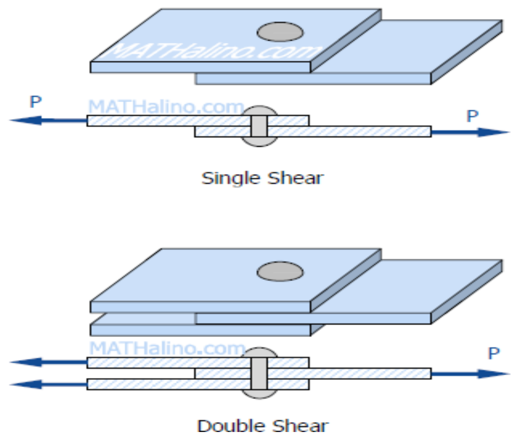


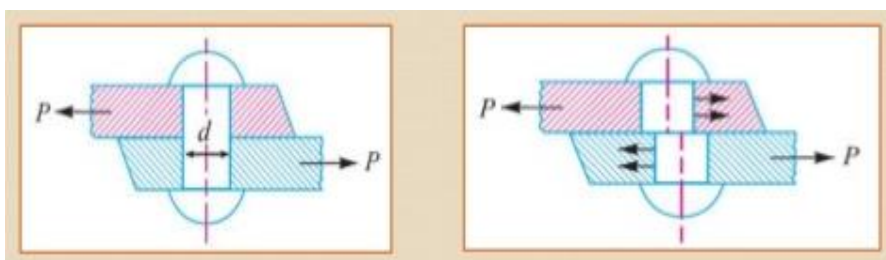
Shear stresses

When a body is subjected to two equal and opposite forces acting tangentially across the resisting section, as a result of which the body tends to shear off the section, then the stress induced is called shear stress (τ), the corresponding strain is called shear strain (Φ).

$$shear\ stress(\tau) = \frac{tangential\ force}{resisting\ area}\ \frac{N}{mm^2}$$



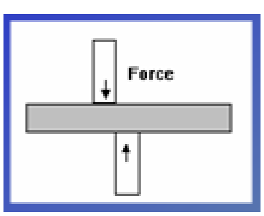




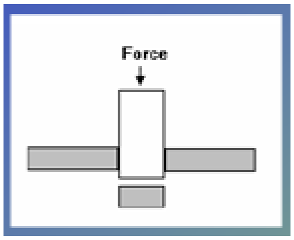
The resulting force intensities are known as shear stresses, the mean shear stress being equal to the corresponding average shear stress (τ)= $\frac{P}{Area}$

Where P is the total force and A the area over which it act

Shear force phenomenon



When a pair of shears cuts a material



When a material is punched

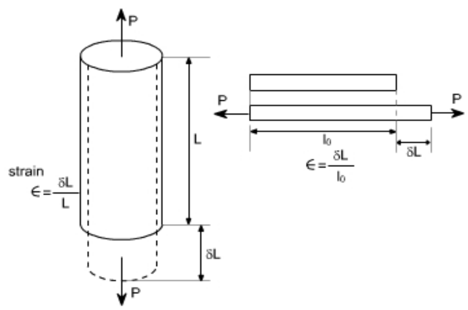
When a beam has a transverse load

The sign convention of shear force and shear stress is based on how it shears the material as shown in the figure.

STRAIN

Concept of strain: change in dimensions of a body to its original dimensions under external loading can be defined as strain.

Consider a bar is subjected to a direct load, and hence a stress will be induced in the bar will subject to Change in length. If the bar has an original length L and changes by an amount δL, the strain produce is defined as follows

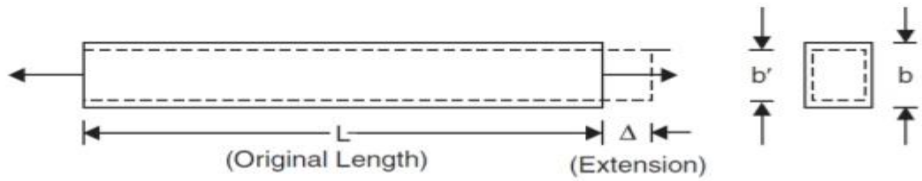


$$strain\ (\epsilon) = \frac{change\ in\ length}{original\ length} = \frac{\delta L}{L}$$

Strain is thus, a measure of the deformation of the material and is a non dimensional Quantity i.e. it has no units. It is simply a ratio of two quantities with the same unit

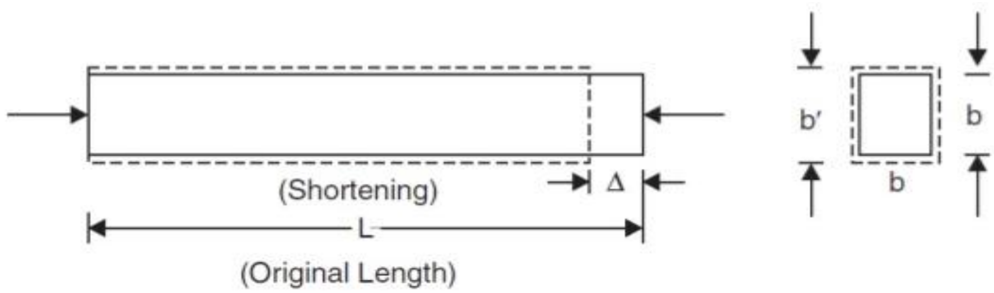
Types of strains

Tensile strain



When a bar is subjected to tensile load there will be decrease in cross sectional area and increase in length of the body. The ratio of increase in length of the body to its original length is called tensile strain.

Compressive strain



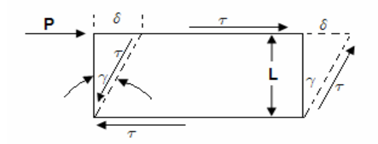
When a bar is subjected to compressive load there will be increase in cross sectional area and decrease in length of the body. The ratio of decrease in length of the body to its original length is called compressive strain.

Tensile strains are positive whereas compressive strains are negative. The strain defined Earlier was known as linear strain or normal strain or the longitudinal strain now let us

Define the shear strain.

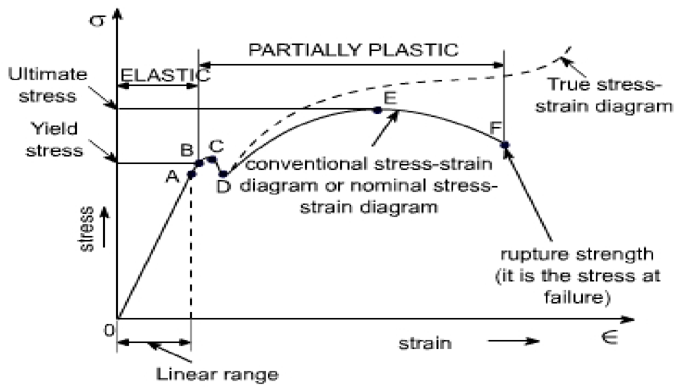
Shear Strain (γ): When a force P is applied tangentially to the element shown. Its edge displaced to dotted line. Where δ_i is the lateral displacement of the upper face of the element relative to the lower face and L is the distance between these faces.

Then the shear strain is $(\gamma) = \dfrac{\delta}{L}$



This shear strain can be defined as the change in right angle or the angle of deformation δ is then termed as the shear strain. Shear strain is measured in radians & hence is non – dimensional i.e. it has no unit.

Stress strain diagram for ductile material



Salient points of the graph

(A) So it is evident from the graph that the strain is proportional to strain or elongation is Proportional to the load giving a straight line relationship. This law of proportionality is valid up to a point A. or we can say that point A is some ultimate point when the linear nature of the graph ceases or there is a deviation from the linear nature. This point is known as **the limit of proportionality or the proportionality limit**.

(B) For a short period beyond the point A, the material may still be elastic in the sense that the deformations are completely recovered when the load is removed. The limiting point B is termed as **Elastic Limit** .

(C) and (D) - Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation or permanent set when load is removed. These two points are termed as upper and lower yield points respectively. The stress at the yield point is called the yield strength.

- Study a stress – strain diagrams shows that the yield point is so near the proportional limit. For most of the cases two may be taken as one. However, it is much easier to locate the former. For material which does not possess a well define yield points, In order to find the yield point or yield strength, an offset method is applied.

- In this method a line is drawn parallel to the straight line portion of initial stress diagram by offsetting this by an amount equal to 0.2% of the strain as shown as below and this happens especially for the low carbon steel.

(E) A further increase in the load will cause marked deformation in the whole volume of the metal. The maximum load which the specimen can with stand without failure is called the load at the ultimate strength. The highest point ‘E' of the diagram corresponds to the ultimate strength of a material. S_u= Stress which the specimen can with stand without failure & is known as Ultimate Strength or Tensile Strength. S_u is equal to load at E divided by the original cross-sectional area of the bar.

(F) Beyond point E, the bar begins to forms neck. The load falling from the maximum until fracture occurs at F.

[Beyond point E, the cross-sectional area of the specimen begins to reduce rapidly over a relatively small length of bar and the bar is said to form a neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F]

Note: Owing to large reduction in area produced by the necking process the actual stress at fracture is often greater than the above value. Since the designers are interested in maximum loads which can be carried by the complete cross section, hence the stress at fracture is seldom of any practical value.

Percentage Elongation

The ductility of a material in tension can be characterized by its elongation and by the reduction in area at the cross section where fracture occurs. It is the ratio of the extension in length of the specimen after fracture to its initial gauge length, expressed in percent.

$$\delta = \frac{l_1 - l_g}{l_g} \text{ x100}$$

l_1 = gauge length of specimen after fracture (or the distance between the gage marks at fracture)

lg= gauge length before fracture(i.e. initial gauge length)

For 50 mm gage length, steel may here a % elongation d of the order of 10% to 40%.

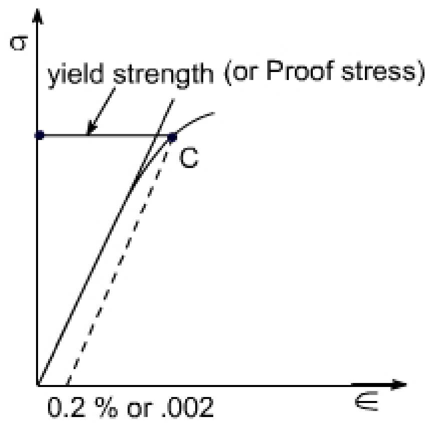
Percentage reduction in area:

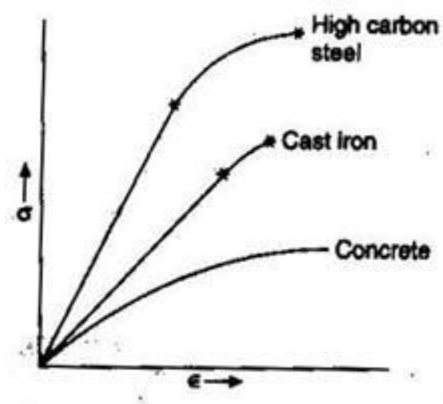
$$A\% = \frac{A_o - A_f}{A_o} \text{ X } 100$$

A_o = original cross sectional area

A_f = final cross sectional area after deformation

Stress strain diagram for brittle material





Hooke's law

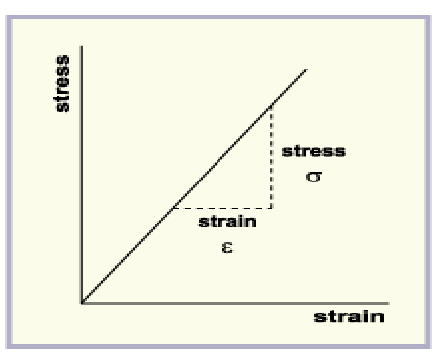
According to Hooke's law the stress is directly proportional to strain up to proportionality limit. i.e. normal stress (σ) α normal strain (ε) and shearing stress (τ) α shearing strain (γ)

$$\sigma = E\varepsilon \text{ and } \tau = G\gamma$$

The co-efficient E is called the **modulus of elasticity or Young's modulus** i.e. its resistance to elastic strain.

$$E = \sigma / \varepsilon \quad N/mm^2$$

The coefficient G is called the **shear modulus of elasticity** or **modulus of rigidity.**



Hook's law

$$E = \frac{stress}{strain} = \frac{\sigma}{\epsilon}$$

$$E = \frac{\frac{P}{A}}{\frac{\delta L}{L}}$$

The value of Young's modulus E is generally assumed to be the same in tension or compression and for most engineering material has high, numerical value of the order of 200 Gpa

Working stress

The working stress or allowable stress is the maximum safe stress a material may carry.

The working stress should not exceed proportional limit. Since the proportional limit is difficult to determine accurately, we take yield point or the ultimate strength and divide this stress by a suitable number N, called the factor of safety. Thus,

$$\sigma_W \text{ (Working stress)} = \sigma_{\gamma p} / F_{\gamma p}$$

$$\sigma_W = \sigma_{ult} / F_{ult}$$

We use yield point stress for calculating σ_w for structural steel design.

Factor of safety

Factor of safety, also known as (and used interchangeably with) **safety factor** (SF), is a term describing the capacity of a system beyond the expected loads or actual loads. Essentially, the **factor of safety** is how much stronger the system is than it usually needs to be for an intended load.

It is the ratio of ultimate stress of a material to the working stress of the same material

$$F.S = \frac{\text{ultimate stress}}{\text{working stress}}$$

Ultimate stress is the stress of a material that can be observed to the maximum extent.

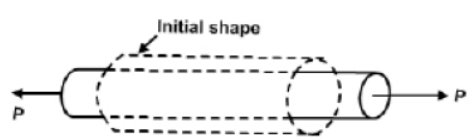
For mild steel factor of safety is

$$F.S = \frac{\text{yield stress}}{\text{working stress}}$$

Poisson's Ratio (μ): it is the ratio of lateral strain to the longitudinal strain

In case of a circular bar lateral strain is change in diameter to its original diameter and longitudinal strain is change in length to its original length when the bar subjected to axial loading.

$$= \frac{\text{Transverse strain or lateral strain}}{\text{Longitudinal strain}} = \frac{-\epsilon_y}{\epsilon_x}$$



(Under unidirectional stress in *x*-direction)

Poisson's ratio in various materials

Material	Poisson's ratio	Material	Poisson's ratio
Steel	0.25 – 0.33	Rubber	0.48 – 0.5
C.I	0.23 – 0.27	Cork	Nearly zero
Concrete	0.2	Novel foam	negative

It has been observed that for elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio.

For most of the engineering materials the value lies between 0.25 and 0.33.

Elongation of a uniform bar

For a prismatic bar loaded in tension by an axial force P as shown in below figure. The elongation of the bar can be determined as

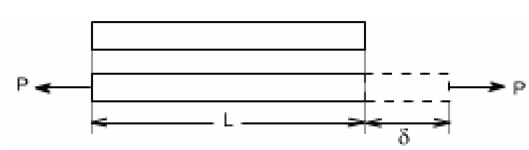
We know that Young's modulus is the ratio of longitudinal stress to the longitudinal strain

$$E = \frac{\sigma}{\varepsilon}$$

$Stress\ \ \sigma = \frac{P}{A}\ \ \ N/mm^2$

$Strain\ \ \varepsilon = \frac{\delta}{L}$

δ is change in length of the bar $\delta = \dfrac{PL}{AE}$ mm

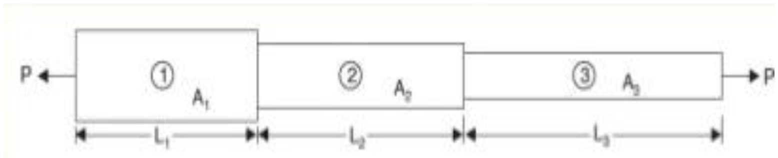


Elongation of bar having varying cross section

A bar having different cross-sections and subjected to axial load P. length of three portions L_1,L_2 and L_3 and respective cross sectional areas are A_1,A_2 and A_3

E is the young's modulus of the material in N/mm^2

P is applied axial load in Newtons

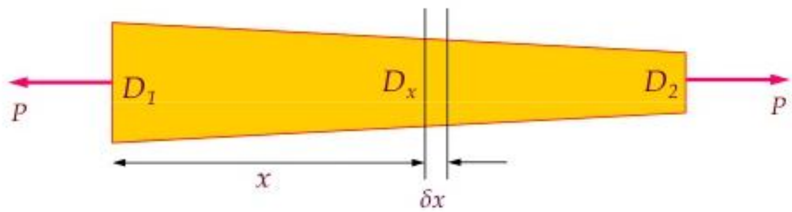


Total Elongation of a bar

$$\delta = \frac{P}{E}\left(\frac{L_1}{A_1} + \frac{L_2}{A_2} + - - - - - - - + \frac{L_n}{A_n}\right)\ mm$$

Elongation of a taper bar subjected to axial loading

A uniform varying taper bar having bigger diameter D_1 and smaller diameter D_2 and length L subjected to an axial pull P. Take E is the young's modulus of the material, then the elongation is



$$D_x = D_1 - \left(\frac{D_1 - D_2}{L}\right)x$$

$$\Delta = \frac{P\delta x}{\left(\frac{\pi D_x^2}{4}\right)E}$$

$$\therefore \text{Total elongation, } \Delta L = \int_0^L \frac{P\delta x}{\left(\frac{\pi D_x^2}{4}\right)E} = \frac{4PL}{\pi E D_1 D_2}$$

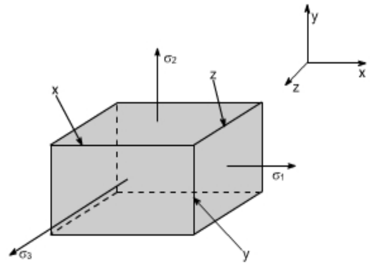
$$\text{When } D_1 = D_2 = D,\ \Delta L = \frac{4PL}{\pi E D^2}$$

$$\text{Elongation } \Delta L = \delta L = \frac{4PL}{\pi E\, D_1 D_2}$$

Volumetric Strain

When a three dimensional body subjected to three mutually perpendicular stress increase or decrease in volume takes place based on direction of applied forces as shown in the figure. Volumetric strain can be defined ad change in volume of the body to its original volume.

Consider a body having Young's modulus E and poisons ratio μ subjected to three mutually perpendicular stress as shown in figure.



From the above figure stress in x-direction is σ_1 and strain in corresponding direction is ε_1.

Then strain due to σ_1 x-direction = σ_1/E

strain due to σ_2 x-direction σ_1 = $-\mu\, \sigma_2/E$

strain due to σ_3 x-direction σ_1 = $-\mu\, \sigma_3/E$

thus net strain in the direction of σ_1, $\varepsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \frac{\sigma_3}{E}$

in similar way $\varepsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \frac{\sigma_3}{E}$

$$\varepsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \frac{\sigma_2}{E}$$

Volumetric strain is $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \frac{\sigma_3}{E} + \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \frac{\sigma_3}{E} + \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \frac{\sigma_3}{E}$

$$= \frac{\sigma_1 + \sigma_2 + \sigma_3}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$

$$volumetric\ strain = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

ELASTIC CONSTANTS

In considering the elastic behavior of isotropic materials under, normal, shear and hydrostatic loading, we introduce a total of four elastic constants namely E, G, K an **μ**. It turns out that not all of these are independent to the others. In fact, given any two of them, the other two can be find out . Let us define these elastic constants

(i) E = Young's Modulus of Rigidity

E= normal Stress / normal strain

(ii) G = Shear Modulus or Modulus of rigidity

G= Shear stress / Shear strain

(iii) **μ** = Possion's ratio

μ = lateral strain / longitudinal strain

(iv) K = Bulk Modulus of elasticity

K= Volumetric stress (direct stress)/ Volumetric strain

Where

Volumetric strain = sum of linear strains in x, y and z direction.

Volumetric stress = stress which cause the change in volume.

RELATION AMONG ELASTIC CONSTANTS

$$E=2G(1+\mu)$$

$$E=3K(1-2\mu)$$

$$E=\frac{9KG}{3K+G}$$

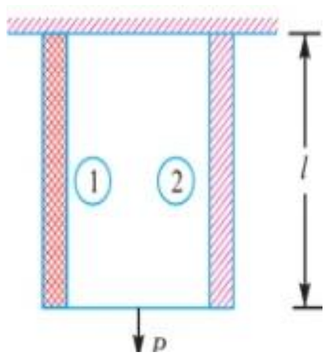
COMPOSITE BARS

- A composite bar made of two bars of different materials rigidly fixed together so that both bars strain together under external load.
- Since strains in the two bars are same, the stresses in the two bars depend on their Young's modulus of elasticity.

In certain application it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. In over head electric cables or Transmission Lines for example it is often convenient to carry the current in a set of copper wires surrounding steel wires. The later being designed to support the weight of the cable over large spans. Such a combination of materials is generally termed compound bars.

Consider two bars of different materials suspended at one end and loaded at free end in this case.

1. The extension or contraction of the bar I being equal i.e strain in the both members is same.
2. The total external load on the bar is equal to the sum of loads carried by different material.



P_1 = Load carried by bar 1,
A_1 = cross sectional area of bar 1,
σ_1 = stress produced in bar 1,
E_1 = Young's modulus of bar 1,
P_2, A_2, σ_2, E_2 are corresponding values of bar 2,
P = total load on the bar,
l = length of the composite bar,
δl is the elongation of the composite bar

We know that $P = P_1 + P_2$

Stress in bar 1 is $\sigma_1 = \frac{P_1}{A_1}$ N/mm^2

Strain in bar 1 is $\epsilon_1 = \frac{\sigma_1}{E_1} = \frac{P_1}{A_1 E_1}$

Elongation in bar -1 $\delta l_1 = \frac{P_1 l}{A_1 E_1}$ mm

Elongation in bar -2 $\delta l_2 = \frac{P_2 l}{A_2 E_2}$ mm

We know $\delta l_1 = \delta l_2$ so $\frac{P_1 l}{A_1 E_1} = \frac{P_2 l}{A_2 E_2}$

Therefore $\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$

$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$

AE is the axial rigidity and $\frac{E_1}{E_2}$ is called modular ratio.

THERMAL STRESSES

Ordinary materials expand when heated and contract when cooled, hence, an increase in temperature produce a positive thermal strain. Thermal strains usually are reversible in a sense that the member returns to its original shape when the temperature return to its original value. However, there are some materials which do not behave in this manner.

When a material undergoes a change in temperature, it either elongates or contracts depending upon whether temperature is increased or decreased of the material.

If the elongation or contraction is not restricted, i. e. free then the material does not experience any stress despite the fact that it undergoes a strain.

The elongation of a bar of length l due to raise in temperature T and having co-efficient of thermal expansion α is

$$\delta l = l\alpha T$$

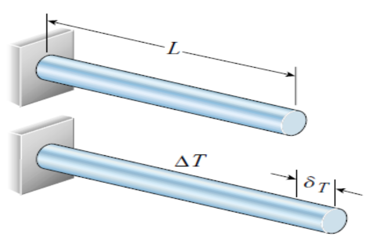
The strain due to temperature change is called thermal strain and is expressed as,

$$\varepsilon = \alpha\ T$$

Where α is co-efficient of thermal expansion, a material property, and T is the change in temperature. The free expansion or contraction of materials, when restrained induces stress in the material and it is referred to as thermal stress.

$\sigma = E\ \alpha\ T$ Where, E = Modulus of elasticity

Thermal stress produces the same effect in the material similar to that of mechanical stress. A compressive stress will produce in the material with increase in temperature and the stress developed is tensile stress with decrease in temperature.



$$\delta l = l\alpha T$$

α = coefficient of linear expansion for the material

L = original Length
T = temp. Change

If however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. They stress is equal in magnitude to that which would be produced in the bar by initially allowing the bar to its free length and then applying sufficient force to return the bar to its original length.

Temperature stresses are developed if a material is prevented from expansion which is in case a support yields or is unable to prevent the expansion completely, then if the yield is ‘a’ then

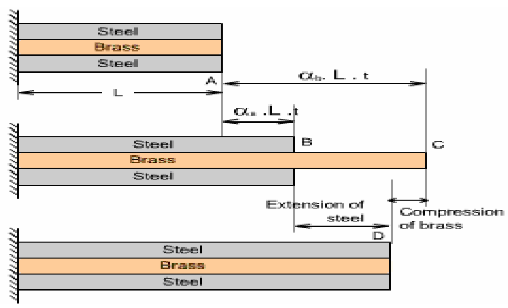
$$\delta l = (l\alpha T - a) = \frac{\sigma\, L}{E}$$

$$\sigma = \frac{(l\alpha T - a)E}{L}$$

Thermal stress on Brass and Mild steel combination

A brass rod placed within a steel tube of exactly same length. The assembly is making in such a way that elongation of the combination will be same. To calculate the stress induced in the brass rod, steel tube when the combination is raised by t°C then the following analogy has to do.

(a) Original bar before heating.
(b) Expanded position if the members are allowed to expand freely and independently after heating.
(c) Expanded position of the compound bar i.e. final position after heating.



In this case both steel and brass subjects to free expansion due to raise in temperature t then

Free expansion of steel is $\delta_{st} = L_s\, \alpha_s t$

Free expansion of brass is $\delta_{bt} = L_b\, \alpha_b t$

Because of adhesive bonding between these two bars steel subjects to tensile force and brass subjects to compressive force. The elongation due to these forces can be written as

Elongation due to Tension in steel bar $\delta_{sf} = \frac{P_s L_s}{A_s E_s}$

Contraction due to compression in brass bar $\delta_{bf} = \frac{P_b L_b}{A_b E_b}$

Compatibility Equation:

$$\delta = \delta_{st} + \delta_{sf} = \delta_{bt} - \delta_{bf}$$

$$\delta = L_s\, \alpha_s t + \frac{P_s L_s}{A_s E_s} = L_b\, \alpha_b t - \frac{P_b L_b}{A_b E_b}$$

• Equilibrium Equation:

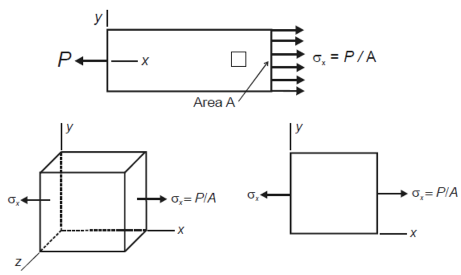
$$\sigma_s A_s = \sigma_b A_b$$

Where, δ = Expansion of the compound bar = AD in the above figure.
δ_{st} = Free expansion of the steel tube due to temperature rise to C = $\alpha_s L t$
= AB in the above figure.
δ_{st} = Expansion of the steel tube due to internal force developed by the unequal expansion.
= BD in the above figure.
δ_{bt} = Free expansion of the brass rod due to temperature rise t°C = $\alpha_b L t$
= AC in the above figure.
δ_{bt} = Compression of the brass rod due to internal force developed by the unequal expansion.
= BD in the above figure.
Tensile force in the steel tube = Compressive force in the brass rod
Where,
σ_s = Tensile stress developed in the steel tube.
σ_B = Compressive stress developed in the brass rod.
A_s = Cross section area of the steel tube.
A_B = Cross section area of the brass rod.

PRINCIPAL STRESSES

Introduction to stress elements

- Stress elements are a useful way to represent stresses acting at some point on a body as show in fig-1.
- Isolate a small element and show stresses acting on all faces. Dimensions are "infinitesimal", but are drawn to a large scale.
- **Maximum stresses on a bar in tension shown in fig-2**



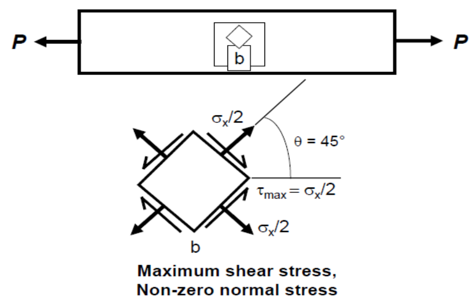
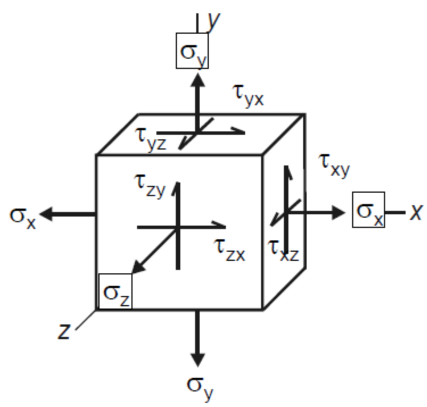


Fig-1

Fig-2

<u>Stresses In three dimension Plane</u>



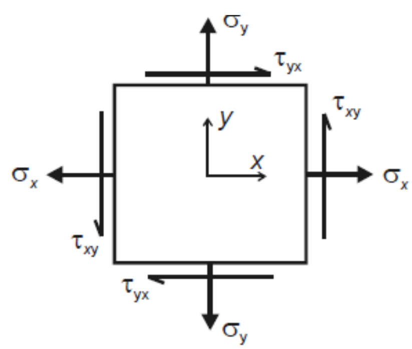
- σ_x, σ_y, σ_z = Positive Normal tensile stresse,
- Shear stresses $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

- Plane stress in the xy plane, only the x and y faces are subjected to stresses ($\sigma_z = 0$ and $\tau_{zx} = \tau_{xz} = \tau_{zy} = \tau_{yz} = 0$).

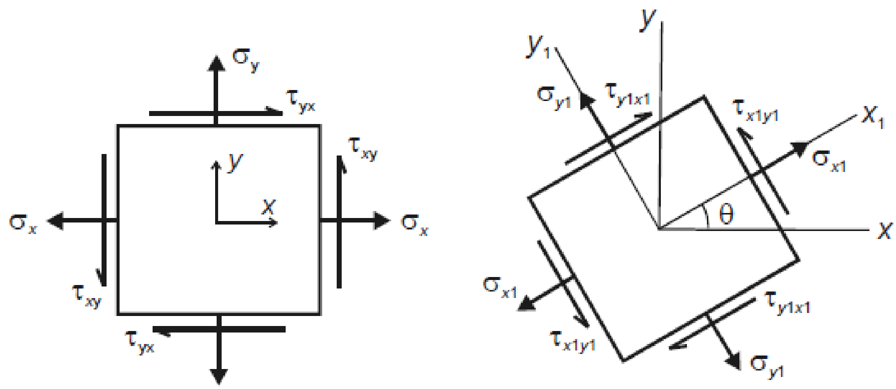
σ_x, σ_y

$\tau_{xy} = \tau_{yx}$

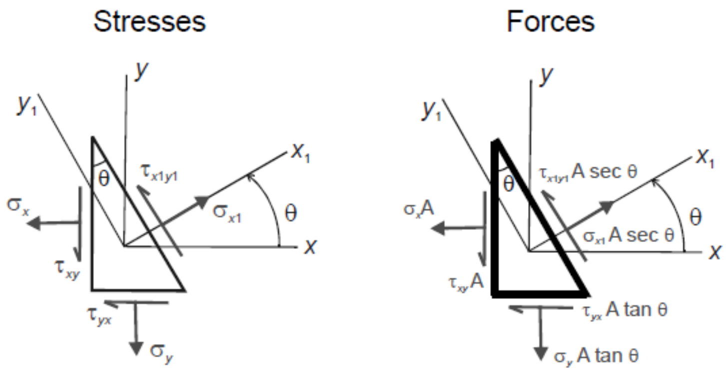
$\sigma_z = 0$



Stresses on Inclined Sections



- The stress system is known in terms of coordinate system *xy*.
- We want to find the stresses in terms of the rotated coordinate system x_1y_1.



- Forces can be found from stresses if the area on which the stresses act is known.
- Force components can then be summed.
- Left face has area A.
- Bottom face has area A tan θ.
- Inclined face has area A sec θ.
- Sum forces in the x_1 direction:
- $$\sigma_{x1}A\sec\theta-(\sigma_xA)\cos\theta-(\tau_{xy}A)\sin\theta-(\sigma_yA\tan\theta)\sin\theta-(\tau_{yx}A\tan\theta)\cos\theta=0$$
  Sum forces in the y_1 direction:
- $$\tau_{x1y1}A\sec\theta+(\sigma_xA)\sin\theta-(\tau_{xy}A)\cos\theta-(\sigma_yA\tan\theta)\cos\theta-(\tau_{yx}A\tan\theta)\sin\theta=0$$
  Using $\tau_{xy}=\tau_{yx}$ and simplifying gives:
  $$\sigma_{x1}=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+2\,\tau_{xy}\sin\theta\cos\theta$$
- $$\tau_{x1y1}=-(\sigma_x-\sigma_y)\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta)$$
  Using the following trigonometric identities

$$\cos^2\theta=\frac{1+\cos 2\theta}{2}\qquad \sin^2\theta=\frac{1-\cos 2\theta}{2}\qquad \sin\theta\cos\theta=\frac{\sin 2\theta}{2}$$

gives the transformation equations for plane stress

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

For stresses on the face, substitute $\theta + 90^{\circ}$ for θ :

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

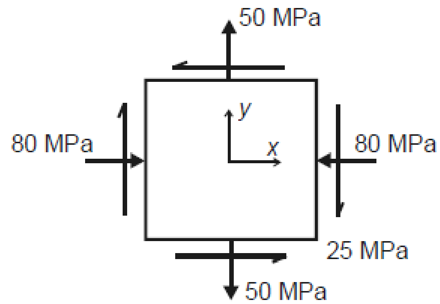
Summing the expressions for and gives :

$$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$$

Can be used to find σ_{y1}, instead of eqn above.

Problem:1

The state of plane stress at a point is represented by the stress element below. Determine the stresses acting on an element oriented 30° clockwise with respect to the original element.



Define the stresses in terms of the established sign convention:
σ_x = -80 MPa σ_y = 50 MPa τ_{xy} = -25 MPa.
We need to find σ_{x1}, σ_{y1}, and τ_{x1y1} when θ = -30°.
Normal Stress along X_1

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x1} = \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(-30°) + (-25)\sin 2(-30°) = -25.9 \text{ MPa}$$

Normal Stress along Y_1

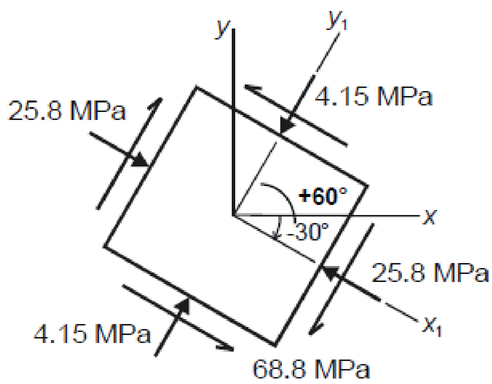
$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_{y1} = \frac{-80+50}{2} - \frac{-80-50}{2}\cos 2(-30°) - (-25)\sin 2(-30°) = -4.15 \text{ MPa}$$

Tangential Stress

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x1y1} = -\frac{(-80-50)}{2}\sin 2(-30°) + (-25)\cos 2(-30°) = -68.8 \text{ MPa}$$



Principal Stresses

- The maximum and minimum normal stresses (σ1 and σ2) are known as the **principal stresses**.
- To find the principal stresses, we must differentiate the transformation equations.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_{x1}}{d\theta} = \frac{\sigma_x - \sigma_y}{2}(-2\sin 2\theta) + \tau_{xy}(2\cos 2\theta) = 0$$

$$\frac{d\sigma_{x1}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

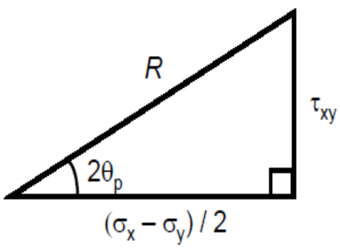
θ_p - principal anglesd

- There are two values of 2θp in the range 0-360°, with values differing by 180°.
- There are two values of θp in the range 0-180°, with values differing by 90°.
- So, the planes on which the principal stresses act are mutually perpendicular

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Principal stresses



$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right)$$

Substituting for *R* and re-arranging gives the larger of the two principal stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

To find the smaller principal stress, use $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$.

$$\sigma_2 = \sigma_x + \sigma_y - \sigma_1 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

These equations can be combined to give:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal planes

The planes on which the principal stresses act are called the **principal planes**. What shear stresses act on the principal planes?

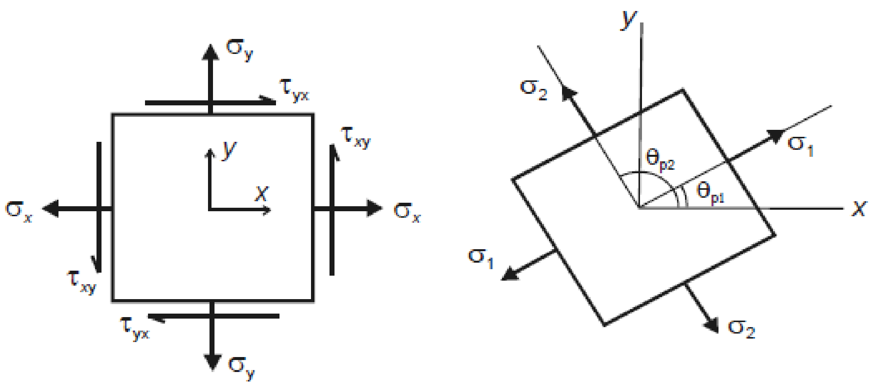
Compare the equations for $\tau_{x1y1} = 0$ and $d\sigma_{x1}/d\theta = 0$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

$$-(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\frac{d\sigma_{x1}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Solving either equation gives the same expression for tan $2\theta_p$ Hence, **the shear stresses are zero on the principal planes**.



- **Principal Stresses**

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

- **Principal Angles** defining the **Principal Planes**

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Methods for determining stress on oblique section

Analytical Method

- This method used to determine normal, tangential and resultant stresses on any oblique planes and position and magnitude of principal stresses.
- It worked based on plane stress transformation equations.

Example: Formulae to calculate stresses when a point in a member subjected in direct stress in one direction.

Normal stress,

$$\sigma_n = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta$$

$$\text{Shear stress, } \tau = \frac{\sigma}{2}\sin 2\theta$$

$$\text{Maximum shearstress, } \tau_{max} = \frac{\sigma}{2}$$

$$\text{Resultant stress, } \sigma_R = \sqrt{[\sigma_n{}^2 + \tau^2]}$$

Problem: A short metallic column of 500 mm2 cross sectional area carries an axial load compressive load of 100 kN. For a plane inclined at 60o with the direction of load, calculate (i) normal stress (ii) Tangential stress (iii) Resultant stress (iv)Maximum shear stress (v) obliquity of resultant stress

Solution

Given data:

$A = 500\ mm^2$

$P = 100\ kN = 100 \times 10^3 N$

$\sigma_x = \frac{P}{A} = \frac{100 \times 10^3}{500} = 200\ N/mm^2$

$\theta = 60^o$ to the direction of load

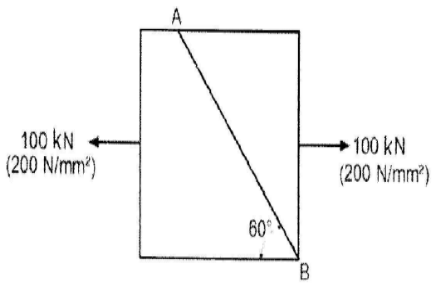
$\sigma_n = ?$

$\tau = ?$

$\sigma_R = ?$

$\tau_{max} = ?$

Angle of obliquity = ?



Calculation of stresses on inclined planes:

Normal stress,

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta = \frac{200}{2} - \frac{200}{2}\cos(2 \times 60°)$$

$$= 100 - 100\cos 120° = 150\ \text{N/mm}^2$$

$$\boxed{\sigma_n = 150\ \text{MPa. Ans}}$$

Shear stress,

$$\tau = \frac{\sigma_x}{2}\sin 2\theta = \frac{200}{2}\sin(2 \times 60°) = 86.6\ \text{N/mm}^2$$

$$\boxed{\tau = 86.6\ \text{MPa. Ans}}$$

Resultant stress,

$$\sigma_R = \sqrt{{\sigma_n}^2 + \tau^2} = \sqrt{150^2 + 86.6^2} = 173\ \text{N/mm}^2$$

$$\boxed{\sigma_R = 173\ \text{MPa. Ans}}$$

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{200}{2} = 100\ \text{N/mm}^2$$

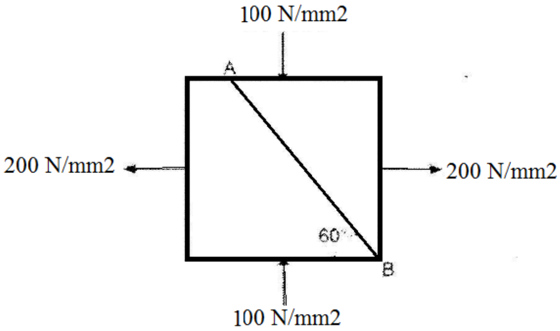
$$\boxed{\tau_{max} = 100\ \text{MPa. Ans}}$$

$$\text{Angle of obliquity} = \tan^{-1}\left[\frac{\tau}{\sigma_n}\right] = \tan^{-1}\left(\frac{86.6}{150}\right)$$

$$\boxed{\text{Angle of obliquity} = 30°.\ \text{Ans}}$$

Member Subjected to Bidirectional stresses:

Problem: The Principal stresses at a point in a bar are 200 N/mm2 (tensile) and 100 N/mm2 (compressive). Determine the resultant stress in magnitude and direction on a plane inclined at 60o to the axis of the major principal stress. Also determine the maximum intensity of shear stress in the material at that point.



$$\text{Normal stress}, \sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

$$= \frac{200 + (-100)}{2} - \frac{200 - (-100)}{2}\cos(2 \times 60°)$$

$$= 125\ \text{N/mm}^2$$

$$\boxed{\sigma_n = 125\ \text{MPa. Ans}}$$

$$\text{Shear stress}, \tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{200 - (-100)}{2}\sin(2 \times 60°)$$

$$= 129.9\ \text{N/mm}^2$$

$$\boxed{\tau = 129.9\ \text{MPa. Ans}}$$

$$\text{Resultant stress, } \sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{125^2 + 129.9^2}$$

$$\boxed{\sigma_R = 180.28 \text{ MPa.} \quad \text{Ans}}$$

Calculation of maximum shearing stress.

$$\tau_{max} = \pm \frac{\sigma_x - \sigma_y}{2} = \frac{200 - (-100)}{2} = \pm 150 \text{ N/mm}^2$$

$$\boxed{\tau_{max} = \pm 150 \text{ MPa. Ans}}$$

<u>A body subjected to two mutually perpendicular Principle tensile stresses accompanied by a simple shear stress.</u>

Problem:

The normal stresses at a point on two mutually perpendicular planes are 140 MPa (tensile) and 100 MPa (compressive). Determine the shear stress on these planes if the maximum principal stress is limited to 150 MPa (tensile). Determine also the following: (i) Minimum principal stress (ii) Maximum shear stress and its plane (iii) Normal, shear and resultant stresses on a plane which is inclined at 30o anti - clockwise to X-X plane

Given data:

$\sigma_x = 140 \text{ N/mm}^2$ (Tensile)

$\sigma_y = -100 \text{ N/mm}^2$ (Compressive)

$\sigma_{p_1} = 150 \text{ N/mm}^2$

$\tau_{xy} = ?$

$\sigma_{p_2} = ?$

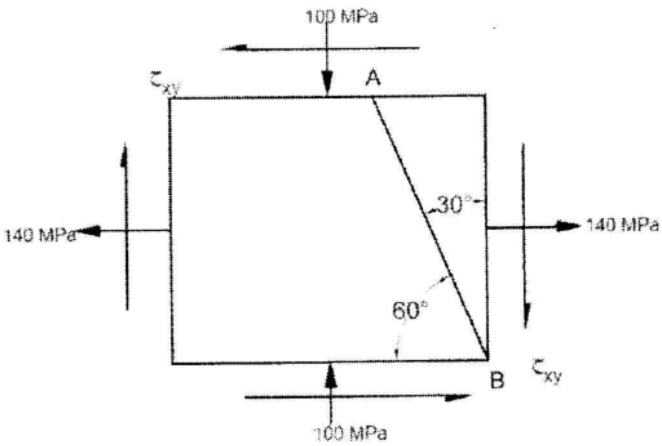
$\tau_{max} = ?$

$\theta_s = ?$

$\theta = 30°$ anticlockwise to the X plane

$= 90° - 30° = 60°$ to the x − x axis

$\sigma_n = ?$



Calculation of Shear stress:

$$\sigma_{p_1} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

$$150 = \frac{140 + (-100)}{2} + \sqrt{\left[\frac{140 - (-100)}{2}\right]^2 + \tau_{xy}^2}$$

$$= 20 + \sqrt{[120]^2 + \tau_{xy}^2}$$

$$130 = \sqrt{[120]^2 + \tau_{xy}^2}$$

$$\tau_{xy} = 50 \text{ N/mm}^2$$

$$\boxed{\tau_{xy} = 50 \text{ MPa. Ans}}$$

Calculation of minor principal stress

$$\sigma_{P2} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

$$= \frac{140 + (-100)}{2} - \sqrt{\left[\frac{140 - (-100)}{2}\right]^2 + 50^2}$$

$$= 20 - 130$$

$$\sigma_{p_2} = -110 \text{ N/mm}^2$$

$$\boxed{\boldsymbol{\sigma_{P2} = 50 \text{ MPa (Compressive). Ans}}}$$

Calculation of maximum shearing stress and their directions:

$$\tau_{max} = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = \sqrt{\left[\frac{140 - (-100)}{2}\right]^2 + 50^2}$$

$$= 130 \text{ N/mm}^2$$

$$\boxed{\boldsymbol{\tau_{max} = 130 \text{ MPa. Ans}}}$$

Let θs = angle which plane of maximum shearing stress makes with $x - x$ axis.

We know that,

$$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{140 - (-100)}{2 \times 50} = 2.4 \quad \text{or}$$

$$2\theta_s = 67.38°$$

$$\boxed{\boldsymbol{\theta_s = 33.7° \quad \text{or} \quad 123.7°. \text{ Ans}}}$$

Calculation of Normal, shear and resultant stresses on a plane which is inclined at 30° anti-clockwise to X plane:

Normal stress,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$= \frac{140 + (-100)}{2} - \frac{140 - (-100)}{2} \cos(2 \times 60°) - 50 \sin(2 \times 60°)$$

$$= 36.7 \text{ N/mm}^2$$

$$\boxed{\boldsymbol{\sigma_n = 36.7 \text{ MPa. Ans}}}$$

Shear stress,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta$$

$$= \frac{140 - (-100)}{2} \sin(2 \times 60°) - 50 \cos(2 \times 60°) = 128.9 \text{ N/mm}^2$$

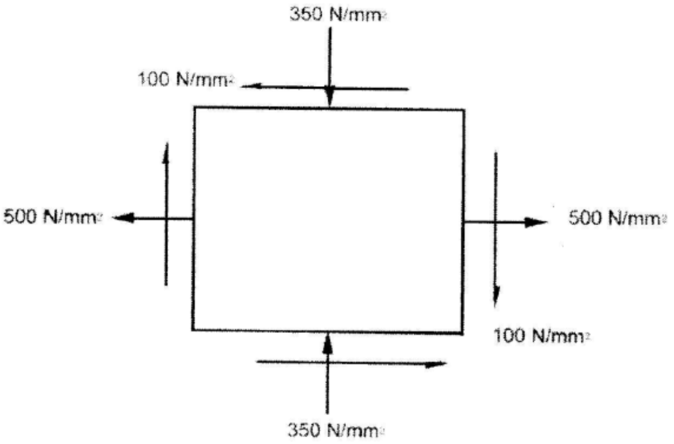
$$\boxed{\boldsymbol{\tau = 128.9 \text{ MPa. Ans}}}$$

Resultant stress,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{36.7^2 + 128.9^2} = 134 \text{ N/mm}^2$$

$$\boxed{\boldsymbol{\sigma_R = 134 \text{ MPa. Ans}}}$$

Problem: An element in a strained material has tensile stress of 500 N/mm^2 and a compressive stress of 350 N/mm^2 acting on two mutually perpendicular planes and equal shear stress of 100 N/mm^2 on these planes. Find the principal stresses and their planes. Find also maximum shear stress and normal stress on the plane of maximum shear stress.



Solution:

Given data:

$\sigma_x = 500\ N/mm^2$ (Tensile)

$\sigma_y = -350\ N/mm^2$ (Compressive)

$\tau_{xy} = 100\ N/mm^2$

$\sigma_{p_1} = ?$

$\sigma_{p_2} = ?$

$\theta_p = ?$

$\tau_{max} = ?$

Normalstress on the plane ofmaximum shearstress, $\sigma_n = ?$

Calculation of Principal stresses and their directions:

$$\sigma_{p_{1,2}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}{}^2}$$

$$= \frac{500 + (-350)}{2} \pm \sqrt{\left[\frac{500 - (-350)}{2}\right]^2 + 100^2}$$

$$= 75 \pm 436.6$$

$$\sigma_{p_1} = 75 + 436.6$$

$$\boxed{\sigma_{p_1} = \mathbf{511.6\ MPa.}\ \ \mathbf{Ans}}$$

$$\sigma_{p_2} = 75 - 436.6$$

$$\boxed{\sigma_{P2} = \mathbf{-361.6\ MPa.}\ \ \mathbf{Ans}}$$

Let θ_p = Angle which plane of principal stress makes with x-x axis

We know that,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 100}{500 - (-350)} = 0.23529$$

$$2\theta_p = 13.24°$$

$$\boxed{\boldsymbol{\theta_p} = \mathbf{6.62°}\ \text{or}\ \mathbf{96.62°}\ \ \mathbf{Ans}}$$

Calculation of maximum shearing stress and their directions:

$$\tau_{max} = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}{}^2} = \sqrt{\left[\frac{500 - (-350)}{2}\right]^2 + 100^2}$$

$$\boxed{\boldsymbol{\tau_{max}} = \mathbf{436.6\ MPa.}\ \ \mathbf{Ans}}$$

Let θp = angle which plane of maximum shearing stress makes with x – x axis.

We know that,

$$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{500 - (-350)}{2 \times 100} = 4.25 \quad \text{or}$$

$$2\theta_s = 76.76°$$

$$\boxed{\theta_s = 38.38° \text{ or } 128.38° \text{ Ans}}$$

Calculation of normal stress on the plane of maximum shearing stress:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{500 + (-350)}{2} - \frac{500 - (-350)}{2}\cos(2 \times 38.38°) - 100 \sin(2 \times 38.38°)$$

$$= 75 - 425\cos 76.76° - 100 \sin 76.76$$

$$\boxed{\sigma_n = -119.68 \text{ MPa. Ans}}$$

Graphical Method

- The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle.
- This graphical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a point in a stresses body.

MOHR'S CIRCLE:

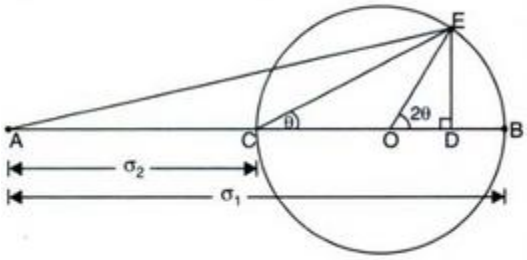
- It is a graphical method to determine normal, tangential and resultant stresses on any oblique planes and position and magnitude of principal stresses.

In Following Cases in Mohr's can use are

A. A body subjected to two mutually perpendicular Principle stresses of unequal intensities
B. A body subjected to two mutually perpendicular Principle stresses of unequal intensities and unlike (I.e one is tensile and other in compressive).
C. A body subjected to two mutually perpendicular Principle tensile stresses accompanied by a simple shear stress.

Case-A (A body subjected to two mutually perpendicular Principle stresses of unequal intensities)

Let



σ_1-Major tensile stress

σ_2-Minor tensile stress

θ-Angle made by oblique plane with axis of minor tensile stress

Drawing Procedure:

1. Any point A draw a horizontal line though A.
2. Draw AB= σ_1 and AC=σ_2 with suitable scale.
3. Describe circle with BC as diameter, center of circle as 'O'
4. Through O draw a line OE making an angle 2θ with OB.
5. Draw ED perpendicular on AB.
6. Join AE.

From figure

Length AD= Normal Stress on oblique plane

Length ED = Tangental stress on oblique plane

Length AE= Resultant stress on oblique plane

Radius of Mohr's circle $= \frac{\sigma 1 - \sigma 2}{2}$

Problem: The tensile stresses at a point across two mutually perpendicular planes are 120N/mm^2 and 60N/mm^2.Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of minor principal stress.

Sol. The data is given

$$\sigma_1 = 120 \text{ N/mm}^2 \text{ (tensile)}$$
$$\sigma_2 = 60 \text{ N/mm}^2 \text{ (tensile)}$$
$$\theta = 30°.$$

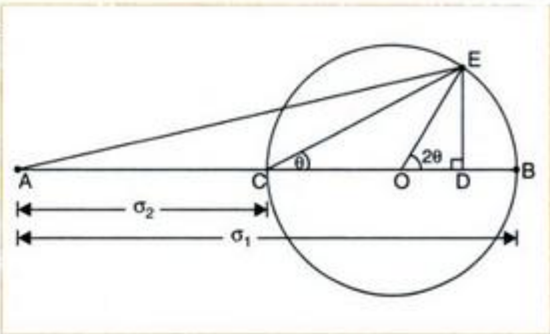
Scale. Let 1 cm = 10 N/mm^2

Then $\sigma_1 = \frac{120}{10} = 12$ cm

and $\sigma_2 = \frac{60}{10} = 6$ cm

Mohr's circle is drawn as : (See Fig. 3.23).

Take any point A and draw a horizontal line through A. Take $AB = \sigma_1 = 12$ cm and $AC = \sigma_2 = 6$ cm. With BC as diameter (*i.e.*, $BC = 12 - 6 = 6$ cm) describe a circle. Let O is the centre of the circle. Through O, draw a line OE making an



angle 2θ (*i.e.*, $2 \times 30 = 60°$) with *OB*. From *E*, draw *ED* perpendicular to *CB*. Join *AE*. Measure lengths *AD*, *ED* and *AE*.

By measurements :

Length $AD = 10.50$ cm

Length $ED = 2.60$ cm

Length $AE = 10.82$ cm

Then normal stress $= \text{Length } AD \times \text{Scale}$

$= 10.50 \times 10 =$ **105 N/mm²**.

Tangential or shear stress $= \text{Length } ED \times \text{Scale}$

$= 2.60 \times 10 =$ **26 N/mm²**.

Resultant stress $= \text{Length } AE \times \text{Scale}$

$= 10.82 \times 10 =$ **108.2 N/mm²**.

<u>CASE-B (A body subjected to two mutually perpendicular Principle stresses of unequal intensities and unlike (I.e one is tensile and other in compressive)).</u>

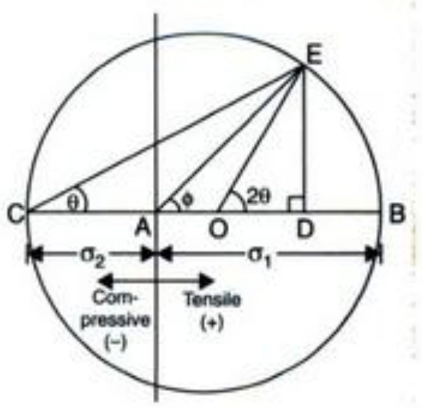
Let

σ_1-Major tensile stress

σ_2-Minor tensile stress

θ-Angle made by oblique plane with axis of minor tensile stress

Procedure:



1. Any point A draw a horizontal line though A.
2. Draw AB= σ_1 (+) towards right of A and AC=σ_2 (+) towards left of A with suitable scale.
3. Bisect BC at O.
4. With O as center and radius equal to OC or OB draw a circle.
5. Through O draw a line OE making an angle 2θ with OB.
6. From E Draw ED perpendicular on AB.
7. Join AE and CE.

From figure

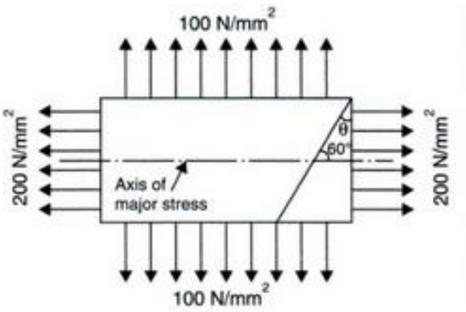
Length AD= Normal Stress on oblique plane

Length ED = Tangental stress on oblique plane

Length AE= Resultant stress on oblique plane

Radius of Mohr's circle $= \frac{\sigma 1+\sigma 2}{2}$

Problem: The stresses at a point in a bar are $200 N/mm^2$(tensile) and $100 N/mm^2$(compressive). Determine the resultant stress in magnitude and direction on a plane inclined at 60 to axis of the major stress. Also determine the maximum intensity of shear stress in the material at the point.



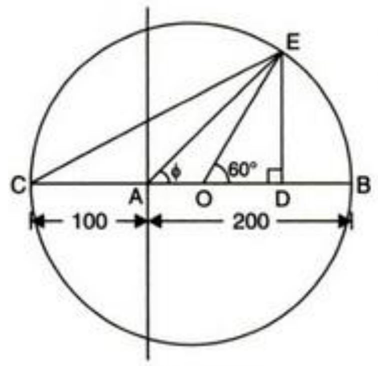
Sol. Given : The data given in problem

$\sigma_1 = 200 \text{ N/mm}^2$

$\sigma_2 = -100 \text{ N/mm}^2$ (compressive)

$\theta = 30°$.

It is required to determine the resultant stress and the maximum shear circle method. First choose a suitable scale.



Let 1 cm represents 20 N/mm^2.

Then $\sigma_1 = \frac{200}{20} = 10$ cm

and $\sigma_2 = \frac{-100}{20} = -5$ cm

By measurement from Fig. , we have

Length $AE = 9.0$ cm

Length $AD = 6.25$ cm and length $ED = 6.5$ cm

Angle $\phi = 46°$

$\therefore$ Resultant stress $=$ Length $AE \times$ Scale

$= 9.0 \times 20 = \mathbf{180 \text{ N/mm}^2}$. **Ans.**

Angle made by the resultant stress with the normal of the inclined plane $= \phi = \mathbf{46°}$. **Ans.**

Normal stress $=$ Length $AD \times 20$

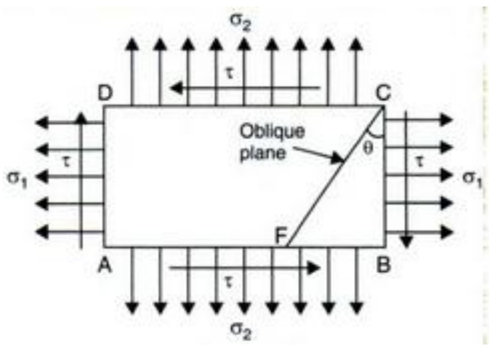
$= 6.25 \times 20 = 125 \text{ N/mm}^2$

Shear stress $=$ Length $ED \times 20$

$= 6.5 \times 20 = 130 \text{ N/mm}^2$.

Case-C (A body subjected to two mutually perpendicular Principle tensile stresses accompanied by a simple shear stress).

Let



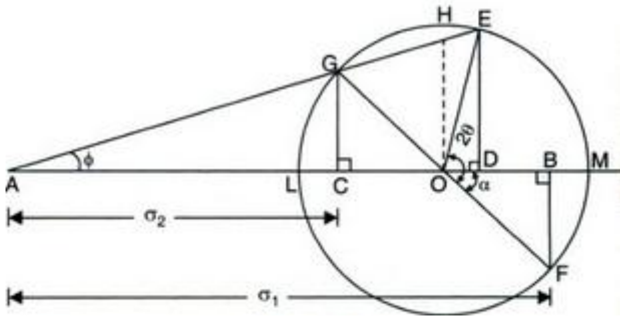
σ_1-Major tensile stress

σ_2-Minor tensile stress

τ- Shear Stress across face BC and AD

θ-Angle made by oblique plane with axis of major tensile stress **Procedure:**

1. Any point A draw a horizontal line though A.
2. Draw AB= σ_1 and AC=σ_2) towards right of A with suitable scale.
3. Draw perpendicular at B and C.
4. Cut off BF and CG equal to shear stress τ with same scale.
5. Bisect BC at O.
6. With O as center and radius equal to OG or OF draw a circle.
7. Through O draw a line OE making an angle 2θ with OF.
8. From E Draw ED perpendicular on CB.
9. Join AE .



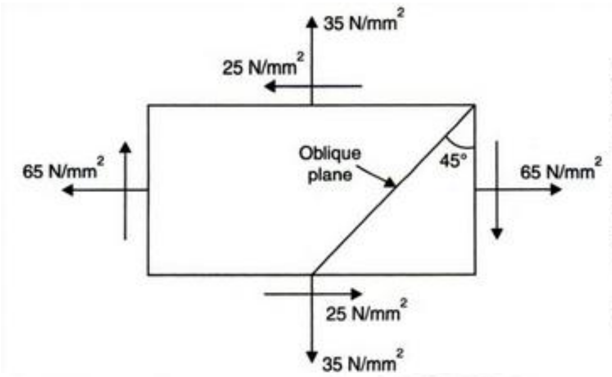
From figure

Length AD= Normal Stress on oblique plane

Length ED = Tangental stress on oblique plane

Length AE= Resultant stress on oblique plane

Problem: A point in a strained material is subjected to Stresses as showing figure below, using Mohr's circle Method; determine the normal and tangential stresses across the oblique plane. Check the answer analytically.



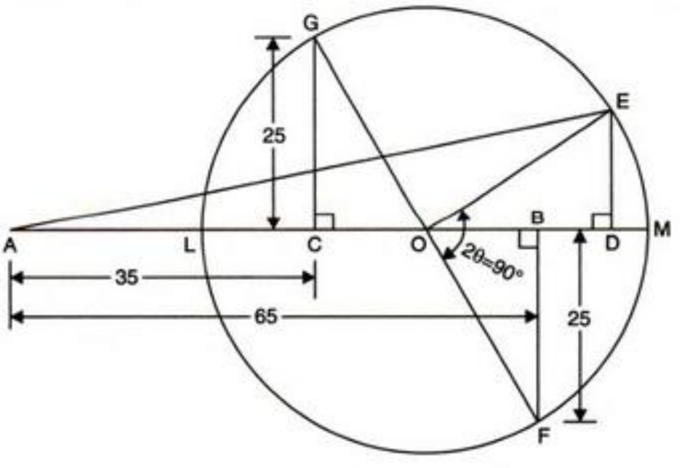
Major principal stress, $\sigma_1 = 65\ N/mm^2$
Minor principal stress, $\sigma_2 = 35\ N/mm^2$
Shear stress, $\tau = 25\ N/mm^2$
Angle of oblique plane, $\theta = 45°$.

Mohr's circle method

Let 1 cm = 10 N/mm^2

Then $\sigma_1 = \frac{65}{10} = 6.5$ cm,

$\sigma_2 = \frac{35}{10} = 3.5$ cm and $\tau = \frac{25}{10} = 2.5$ cm



Mohr's circle

By measurements, length AD = 7.5 cm and

length ED = 1.5 cm.

$\therefore$ Normal stress (σ_n) = Length AD × Scale = 7.5 × 10 = **75 N/mm²**. Ans.

($\because$ 1 cm = 10 N/mm²)

And tangential stress (σ_t) = Length ED × Scale = 1.5 × 10 = **15 N/mm²**. Ans.

Analytical Answers

Normal stress (σ_n) is given by equation (3.12).

$\therefore$ Using equation (3.12),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{65 + 35}{2} + \frac{65 - 35}{2}\cos(2 \times 45°) + 25 \sin(2 \times 45°)$$

$$= 50 + 15 \cos 90° + 25 \sin 90°$$

$$= 50 + 15 \times 0 + 25 \times 1 \qquad (\because \cos 90° = 0, \sin 90° = 1)$$

$$= 50 + 0 + 25 = \mathbf{75\ N/mm^2}.$$ Ans.

Tangential stress is given by equation (3.13)

$\therefore$ Using equation (3.13),

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$$

$$= \frac{65 - 35}{2}\sin(2 \times 45) - 25 \cos(2 \times 45)$$

$$= 15 \sin 90° - 25 \cos 90° = 15 \times 1 - 25 \times 0 = 15 - 0$$

$$= \mathbf{15\ N/mm^2}.$$ Ans.

UNIT II
SHEAR FORCE AND BENDING MOMENT

➢ **Syllabus:**

Shear force and bending moment: Introduction, types of beams, shear force diagrams and bending moment diagrams for cantilever, simply supported and over hanging beams subjected to point loads, uniformly distributed loads, uniformly varying loads, relation between shear force and bending moment.

INTRODUCTION

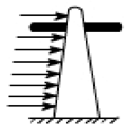
In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These types of members are termed as beams.

There are various ways to define the beams such as
Definition I: A beam is a laterally loaded member, whose cross-sectional dimensions are small as compared to its length.
Definition II: A beam is nothing but a simply a bar which is subjected to forces or couples that lie in a plane containing the longitudinal axis of the bar. The forces are understood to act Perpendicular to the longitudinal axis of the bar.
Definition III: A bar working under bending is generally termed as a beam.

Materials for Beam:
The beams may be made from several usable engineering materials such commonly among them are as follows:

- Metal
- Wood
- Concrete
- Plastic

Examples of Beams:
Refer to the figures shown below that illustrates the beam



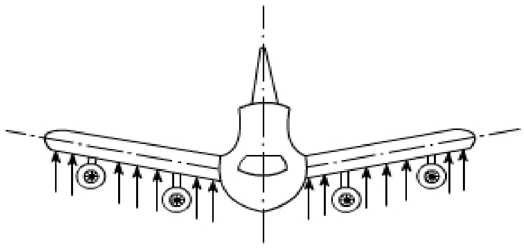


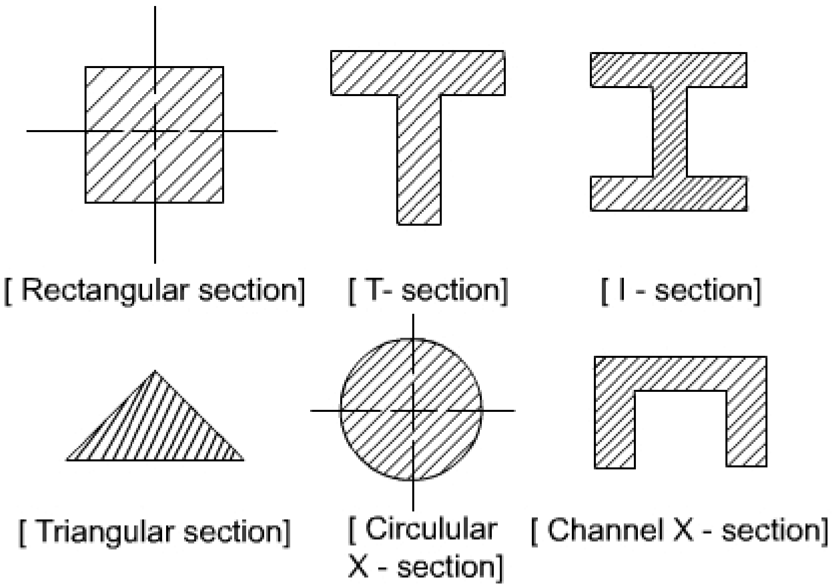
Fig:1(a) Fig:1(b)

- In the fig:1 (a), an electric pole has been shown which is subject to forces occurring due to Wind; hence it is an example of beam.

- In the fig:1(b), the wings of an aeroplane may be regarded as a beam because here the Aerodynamic action is responsible to provide lateral loading on the member

Geometric forms of Beams:

The Area of cross-section of the beam may take several forms some of them have been shown below:



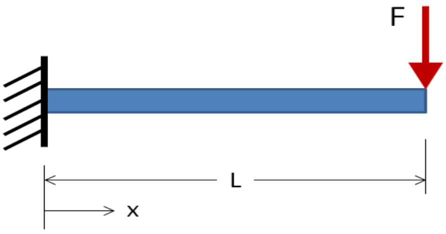
Classification of Beams:

Beams are classified on the basis of their geometry and the manner in which they are supported.

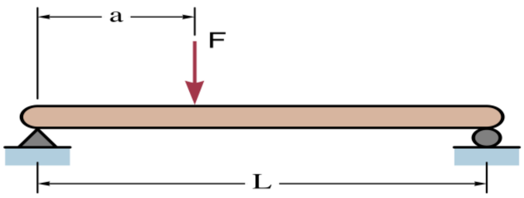
Classification I: The classification based on the basis of geometry normally includes features such as the shape of the cross-section and whether the beam is straight or curved.

Classification II: Beams are classified into several groups, depending primarily on the kind of supports used. But it must be clearly understood why we need supports. The supports are required to provide constraint to the movement of the beams or simply the supports resists the movements either in particular direction or in rotational direction or both. As a consequence of this, the reaction comes into picture whereas to resist rotational movements the moment comes into picture. On the basis of the support, the beams may be classified as follows:

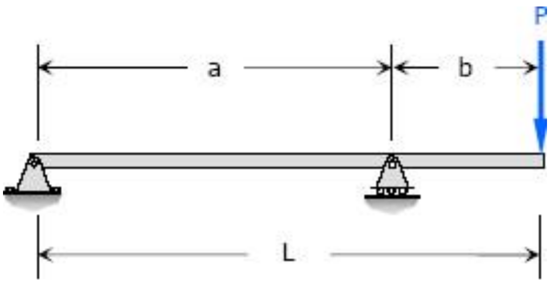
Cantilever Beam: A beam which is supported on the fixed support is termed as a cantilever beam: Now let us understand the meaning of a fixed support. Such a support is obtained by building a beam into a brick wall, casting it into concrete or welding the end of the beam. Such a support provides both the translational and rotational constraint to the beam, therefore the reaction as well as the moments appears, as shown in the figure below



Simply Supported Beam: The beams are said to be simply supported if their supports creates only the translational constraints.
Sometimes the translational movement may be allowed in one direction with the help of rollers and can be represented like this



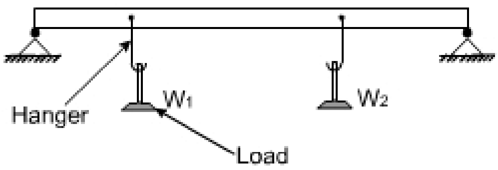
Overhanging beam: If the end portion of a beam is extended beyond the support,such beam is known as overhanging beam.



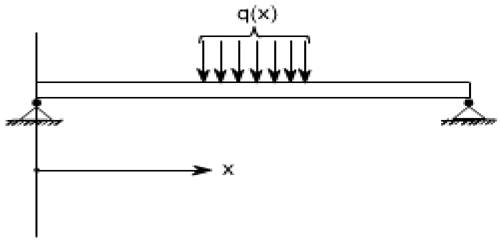
Types of loads act on beams:
A beam is normally horizontal where as the external loads acting on the beams is generally in the vertical directions. In order to study the behaviour of beams under flexural loads, it becomes pertinent that one

must be familiar with the various types of loads acting on the beams as well as their physical manifestations.

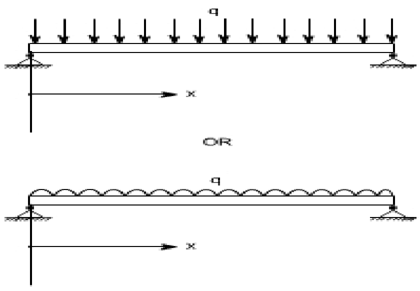
A. Concentrated Load: It is a kind of load which is considered to act at a point. By this we mean that the length of beam over which the force acts is so small in comparison to its total length that one can model the force as though applied at a point in two dimensional view of beam. Here in this case, force or load may be made to act on a beam by a hanger or through other means.



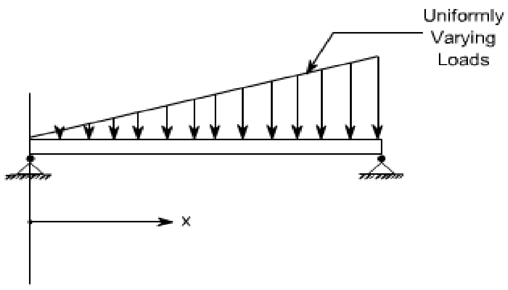
B. Distributed Load: The distributed load is a kind of load which is made to spread over a entire span of beam or over a particular portion of the beam in some specific manner



- In the above figure, the rate of loading ‘q' is a function of x i.e. span of the beam, hence this is a non uniformly distributed load.
- The rate of loading ‘q' over the length of the beam may be uniform over the entire span of beam, then we cell this as a uniformly distributed load (U.D.L). The U.D.L may be represented in either of the way on the beams.

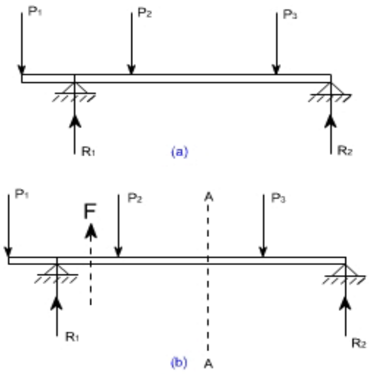


sometimes the load acting on the beams may be the uniformly varying as in the case of dams or on inclined wall of a vessel containing liquid, then this may be represented on the beam as below:



Concept of Shear Force and Bending moment in beams:

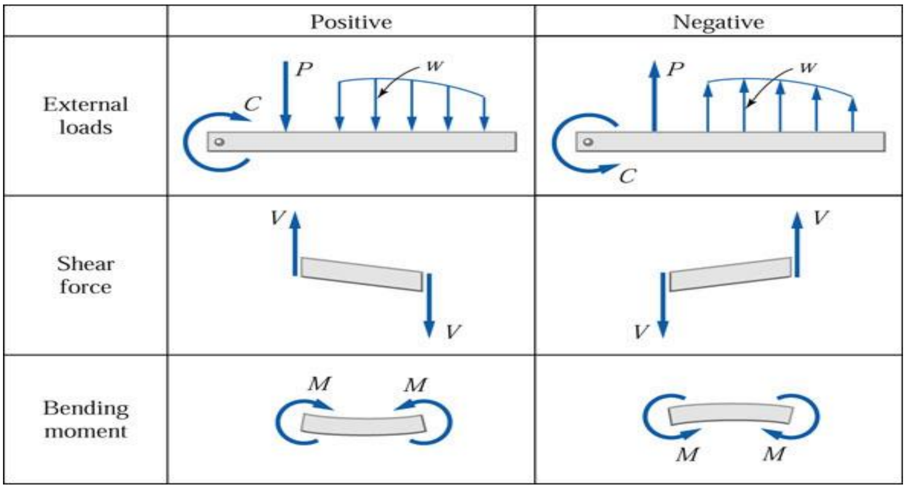
When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further.



- Now let us consider the beam as shown in fig which is supporting the loads P1, P2, P3 and is simply supported at two points creating the reactions R1 and R2 respectively.
- Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA.
- Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards.
- This forces 'F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force 'F' to as follows:
At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention:



Procedure for construction of SF and BM diagrams:

- The positive values of shear force and bending moment are plotted above the base line and negative below the base line.
- If there is vertical load at the section the shear force diagram will increase or decrease suddenly.
- If there is n o loading on the section the shear force diagram will not change
- If there is UDL between two sections the SFD will be an inclined line and BMD will be a curve.
- The bending moment at the free end of a cantilever and at the two ends of simply supported are zeros.
- The BMD always either inclined line or smoother curve.

	Between point loads OR for no load region	Uniformly distributed load	Uniformly varying load
Shear Force Diagram	Horizontal line	Inclined line	Two-degree curve (Parabola)
Bending Moment Diagram	Inclined line	Two-degree curve (Parabola)	Three-degree curve (Cubic-parabola)

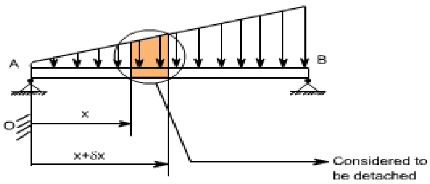
Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further. Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force 'F' varies along the length of beam. If x denotes the length of the beam, then F is function x i.e. F(x). Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment 'M' varies along the length of the beam. Again M is a function x i.e. M(x) respectively.

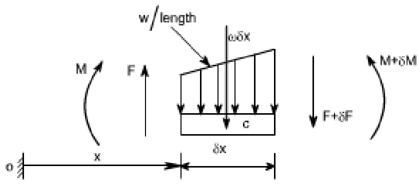
Basic Relationship between the Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

- Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance 'x' from the origin '0'.



- Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ dF at the section x and x + dx respectively
- The bending moment at the sections x and x + dx be M and M + dM respectively.
- Force due to external loading, if 'w' is the mean rate of loading per unit length then the total loading on this slice of length dx is w. dx, which is approximately acting through the centre 'c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre 'c'.
- This small element must be in equilibrium under the action of these forces and couples. Now let us take the moments at the point 'c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M$$ [Neglecting the product of δF and δx being small quantities

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots(2)$$

Conclusions: From the above relations, the following important conclusions may be drawn

- From Equation (1), the area of the shear force diagram between any two points, from
  the basic calculus is the bending moment diagram

$$M = \int F.dx$$

The slope of bending moment diagram is the shear force, thus

$$F = \frac{dM}{dx}$$

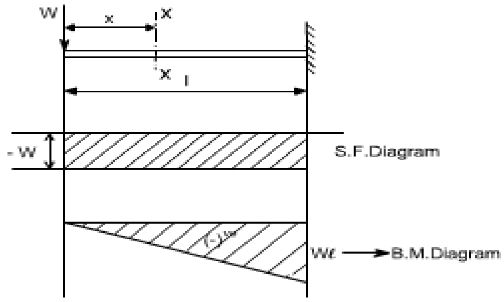
Thus, if F=0; the slope of the bending moment diagram is zero and the bending moment
is therefore constant.
The maximum or minimum Bending moment occurs where

$$\frac{dM}{dx} = 0$$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions.

Construction of shear force and bending moment diagrams:

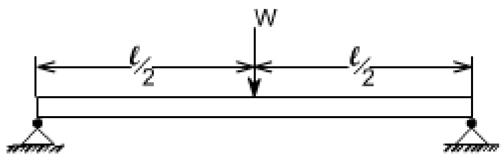
1. **A cantilever of length carries a concentrated load 'W' at its free end.**



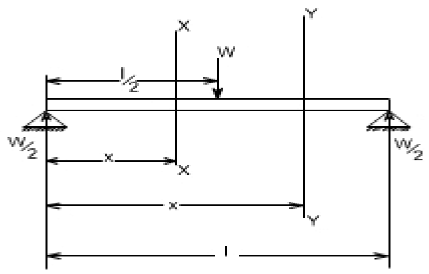
- ❖ At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative..
- ❖ Taking moments about the section gives (obviously to the left of the section) M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention) so that the maximum bending moment occurs at the fixed end i.e. M = -W l.
- ❖ From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W.

The shear force and bending moment are shown above.

2. Simply supported beam subjected to a central load (i.e. Load acting at the mid-way)



- ✓ By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.



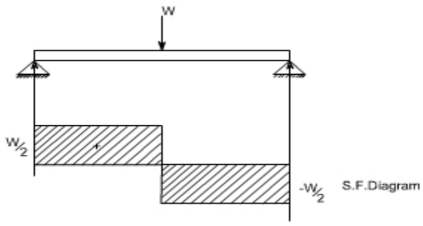
- ✓ So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

  If we consider another section Y-Y which is beyond l/2 then

$$S.F_{Y-Y} = \frac{W}{2} - W = \frac{-W}{2}$$

- ✓ for all values greater = l/2

Hence S.F diagram can be plotted as



For B.M diagram:

If we just take the moments to the left of the cross-section

$$B.M_{x\text{-}x} = \frac{W}{2}x \text{ for x lies between 0 and l/2}$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\frac{l}{2} \text{ i.e B.M at x = 0}$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}x - W\left(x - \frac{l}{2}\right)$$

Again

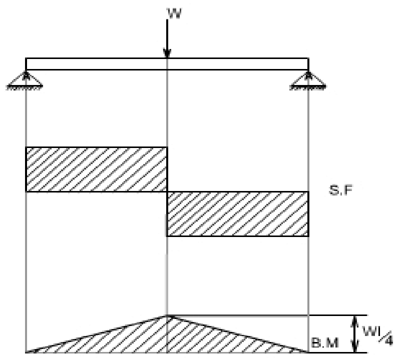
$$= \frac{W}{2}x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}x + \frac{Wl}{2}$$

$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

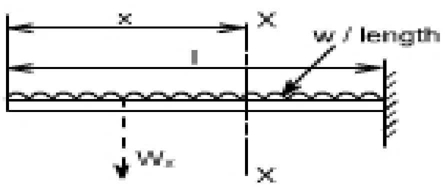
$$= 0$$

Which when plotted will give a straight relation i.e



It may be observed that at the point of application of load there is an abrupt change in the shear force, at this point the B.M is maximum.

2. A cantilever beam subjected to U.d.L, draw S.F and B.M diagram



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

- Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

  S.Fxx = -Wx for all values of 'x'. **---------- (1)**

  S.Fxx = 0

  S.Fxx at x=1 = -Wl

- So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

  Therefore, the bending moment at any cross-section X-X is

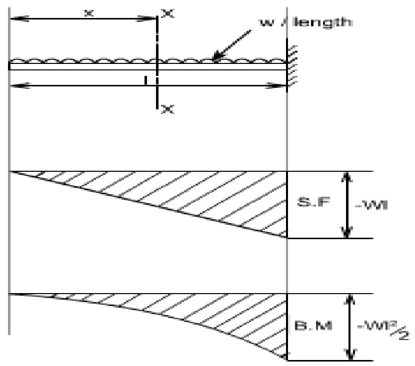
$$B.M_{X-X} = -W \times \frac{x}{2}$$
$$= -W\frac{x^2}{2}$$

- The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation

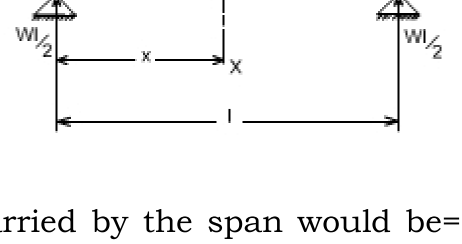
  The extreme values of this would be at x = 0 and x = l

$$B.M_{at\ x = l} = -\frac{Wl^2}{2}$$
$$= \frac{Wl}{2} - Wx$$

Hence S.F and B.M diagram can be plotted as follows:



3. Simply supported beam subjected to a uniformly distributed load [U.D.L].



- The total load carried by the span would be= intensity of loading x length

  = w x l

- By symmetry the reactions at the end supports are each wl/2 If x is the distance of the section considered from the left hand end of the beam.

  S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$

$$= W\left(\frac{l}{2} - x\right)$$

- Giving a straight relation, having a slope equal to the rate of loading or intensity of the loading.

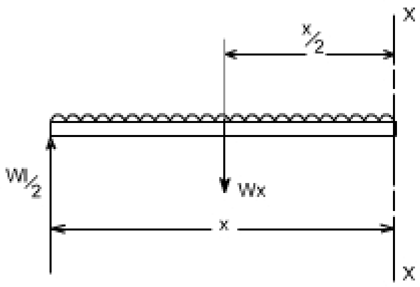
$$S.F_{at\ x=0} = \frac{wl}{2} - wx$$

so at

$$S.F_{at\ x=\frac{l}{2}} = 0 \text{ hence the S.F is zero at the centre}$$

$$S.F_{at\ x=l} = -\frac{Wl}{2}$$

- The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

so the

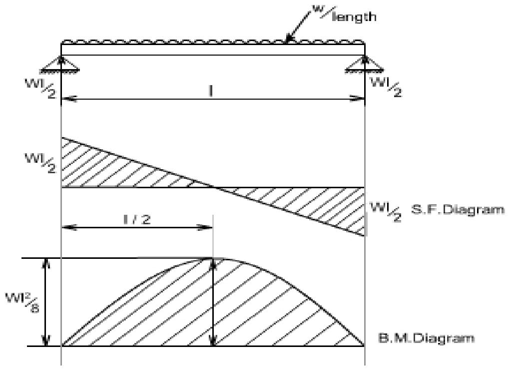
$$= W.\frac{x}{2}(l - 2) \quad \ldots\ldots\ldots(2)$$

$$B.M_{at\ x=0} = 0$$

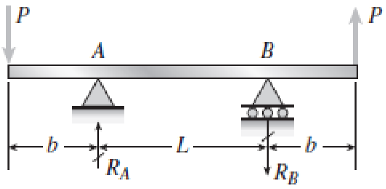
$$B.M_{at\ x=l} = 0$$

$$B.M\Big|_{at\ x=l} = -\frac{Wl^2}{8}$$

- So the equation (2) when plotted against x gives rise to a parabolic curve and the shear force and bending moment can be drawn in the following way will appear as follows:



Overhanging beam with point load:

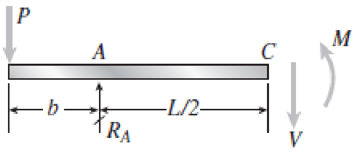


$$\Sigma M_B = 0$$

$$R_A = \frac{1}{L}[P(L+b+b)]$$

$$= P\left(1+\frac{2b}{L}\right) \quad \text{(upward)}$$

$$\Sigma M_A = 0: \quad R_B = P\left(1+\frac{2b}{L}\right) \quad \text{(downward)}$$



FREE-BODY DIAGRAM (C IS THE MIDPOINT)

$$\Sigma F_{\text{VERT}} = 0:$$

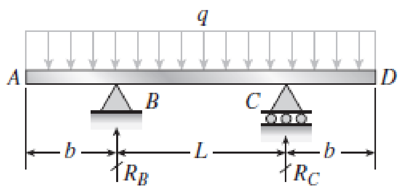
$$V = R_A - P = P\left(1+\frac{2b}{L}\right) - P = \frac{2bP}{L} \quad \leftarrow$$

$$\Sigma M_C = 0:$$

$$M = P\left(1+\frac{2b}{L}\right)\left(\frac{L}{2}\right) - P\left(b+\frac{L}{2}\right)$$

$$M = \frac{PL}{2} + Pb - Pb - \frac{PL}{2} = 0 \quad \leftarrow$$

Overhanging beam with uniformly distributed load:

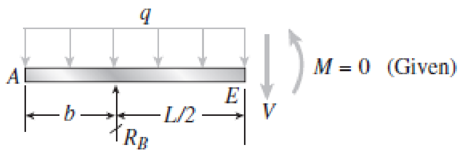


From symmetry and equilibrium of vertical forces:

$$R_B = R_C = q\left(b+\frac{L}{2}\right)$$

FREE-BODY DIAGRAM OF LEFT-HAND HALF OF BEAM:

Point E is at the midpoint of the beam.



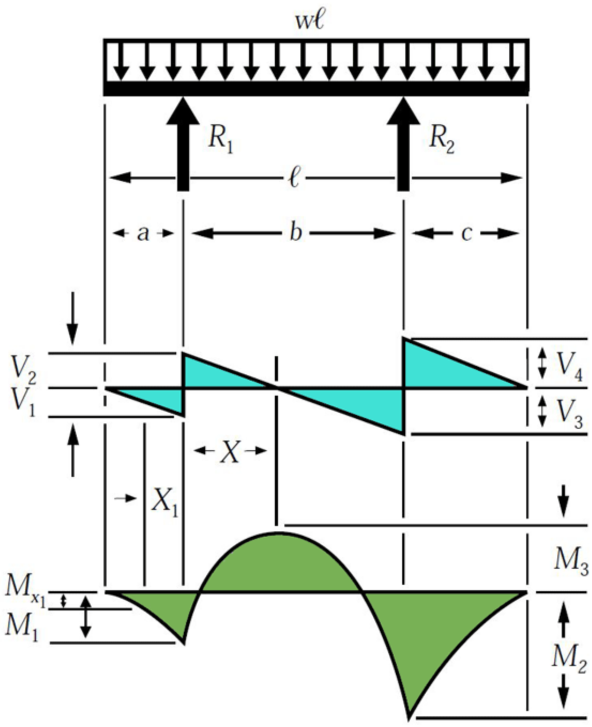
$$\Sigma M_E = 0$$

$$-R_B\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b+\frac{L}{2}\right)^2 = 0$$

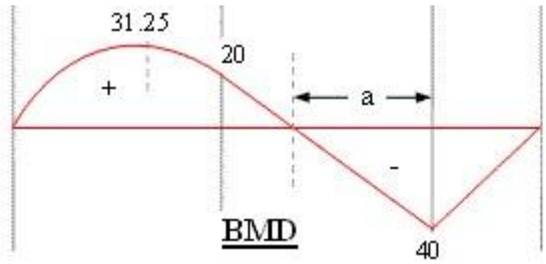
$$-q\left(b+\frac{L}{2}\right)\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b+\frac{L}{2}\right)^2 = 0$$

Solve for b/L :

$$\frac{b}{L} = \frac{1}{2} \quad \leftarrow$$

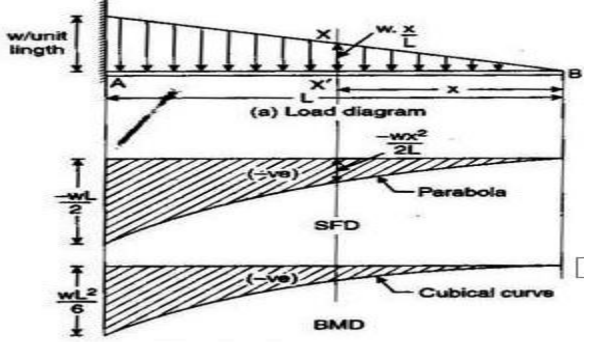


Point of contraflexture



- In a bending beam, a **point** is known as a **point of contraflexure** if it is a location at which no bending occurs.
- In a bending moment diagram, it is the **point** at which the bending moment curve intersects with the zero line.
- In other words where the bending moment changes its sign from negative to positive or vice versa is called point of contraflexture.
- This case can be seen in overhanging beams.

Cantilever subjected to uniformly varying load over whole length



w/L.x = wx/L

Total load from B to X – X' = Area of the load diagram from B to X – X'.

$$= 1/2.x.w.x/L = wx^2/2L$$

Now,

Shear force at section (X – X') 'F_x'

$$= -1/2 \; . \; w.x/L \; . \; x = -wx^2 \; / \; 2L$$

(i.e., Parabola with concave curve)

At x = 0; $F_B = 0$

x = L, FA = –w.L / 2

Bending moment at section (X – X')

$$\text{'}M_x\text{'} = -w.x^2 \; / \; 2L \times 1/3 \; x = - \; w.x^2 \; / \; 6L$$

(i.e., Cubic cubic with concave surface)

At x = 0; $M_B = 0$

At x = L; $M_B = -wL^3 \; / \; 6L = -2L^2 \; / \; 6$

Simply supported beam subjected to UVL

To summarize:

Supports equations: $A_y = \frac{wL}{6}$ $B_y = \frac{wL}{3}$

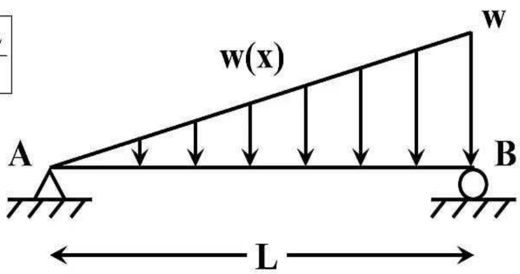
Shear force equation: $V_x = \frac{wL}{2}\left(\frac{1}{3} - \frac{x^2}{L^2}\right)$

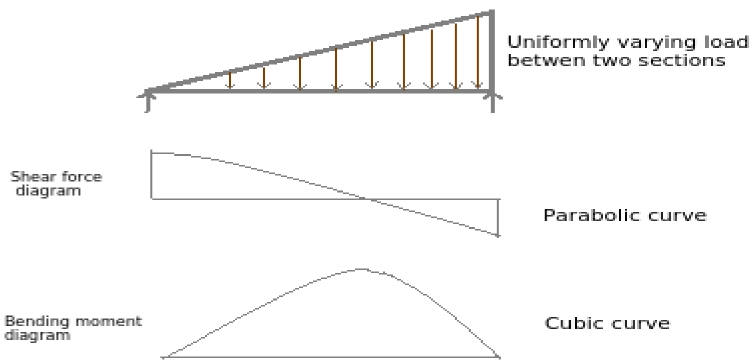
Bending moment equation $M_x = \frac{wLx}{6}\left(1 - \frac{x^2}{L^2}\right)$

Maximum shear force at right support $V_{max} = \frac{wL}{3}$

Minimum shear force at $x = \frac{L}{\sqrt{3}}$

Maximum bending moment at zero shear location $x = \frac{L}{\sqrt{3}}$ and equal to $M_{max} = \frac{wL^2}{9\sqrt{3}}$





SOLID MECHANICS

Unit-III

SOLID MECHANICS

UNIT-III

FLEXTURAL STRESSES

(BENDING STRESSES AND SHEAR STRESSES)

Flextural stresses: Theory of simple bending, Derivation of bending equation, Neutral axis, determination bending stresses, section modulus of rectangular and circular sections, I, T, Angle and channel sections.

Shear stresses: Shear stress equation, Shear stress distribution across various beam sections like rectangular circular, triangular, I, T and Angle sections

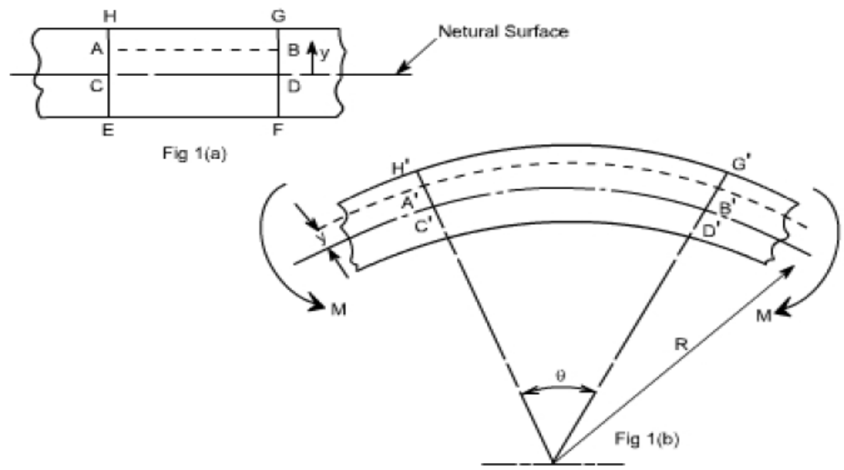
INTRODUCION

- When a beam having an arbitrary cross section is subjected to a transverse loads the beam will bend.
- In addition to bending the other effects such as twisting and buckling may occur.
- to investigate a problem that includes all the combined effects of bending, twisting and buckling could become a complicated one.
- Thus we are interested to investigate the bending effects alone, in order to do so, we have to put certain constraints on the geometry of the beam and the manner of loading.

ASSUMPTIONS

The constraints put on the geometry would form the assumptions

1. Beam is initially straight, and has a constant cross section.
2. Beam is made of homogeneous material and the beam has a longitudinal plane of symmetry.
3. Resultant of the applied loads lies in the plane of symmetry.
4. The geometry of the overall member is such that bending not buckling is the primary cause of failure.
5. Elastic limit is nowhere exceeded and ‘E' is same in tension and compression.
6. Plane cross sections remain plane before and after bending.



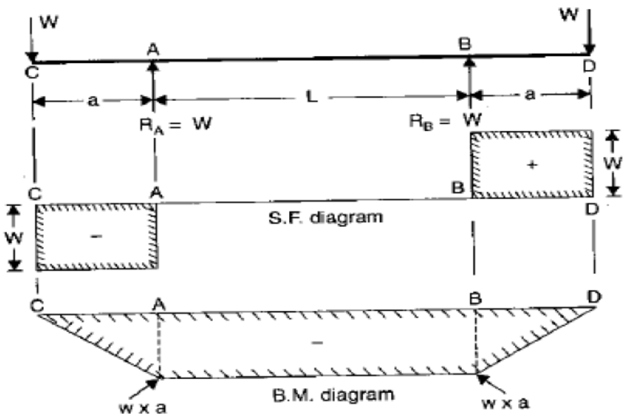
Let us consider a beam initially unstressed as shown in fig 1(a). Now the beam is subjected to a constant bending moment (i.e. ‘Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b) As a result of this bending.

- The top fibers of the beam will be subjected to **tension** and the bottom to **compression.**
- Somewhere between the two extreme fibers i.e between top and bottom layers there are points at which the stress is zero.
- The locus of all such points is known as neutral axis.
- **Neutral plane will not undergo any deformation also it will not be subjected to bending stress.**
- The radius of curvature R is then measured to this axis.
- For symmetrical sections the N. A. is the axis of symmetry but whatever the section N. A. will always pass through the centre of the area or centroid.
- The above restrictions have been taken so as to eliminate the possibility of 'twisting' of the beam.

CONCEPT OF PURE BENDING

As we are aware of the fact internal reactions developed on any crosssection of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant **shear forces are zero** on any crosssection perpendicular to the longitudinal axis of the member.

Since $\frac{dM}{dX} = F = 0$ or M = constant.

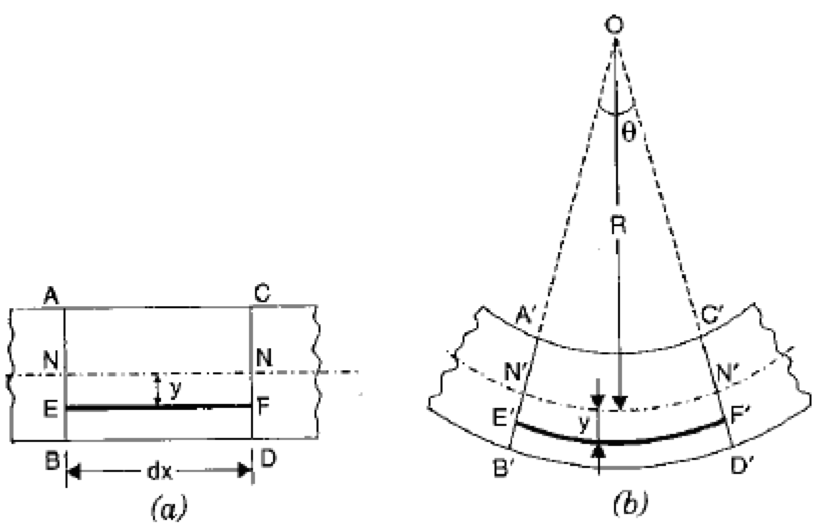


Thus, the zero shear force means that the bending moment is constant or the bending is same at every crosssection of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

DERIVATION FOR BENDING EQUATIONS

If a length of a beam is subjected to a constant bending moment & shear force is zero, then the stresses set up in that length of the beam are known as bending stresses and that length of the beam is said to be in pure bending.

A small length δx of a beam subjected to a simple bending as shown in the figure (a) and due to action of bending, the par of length δ_x will be deformed as shown in the figure (b).



Neutral axis (N-A):

The line of intersection of neutral layer on a cross-section of beam is known as neutral axis.

- Due to the decrease in length of the layers above N-N, these layers will be subjected to compressive stresses.

 Due to the increase in length of the layers above N-N, these layers will be subjected to tensile stresses.

 The amount by which a layer increases or decreases in length, depends upon the position of the layer w.r.t. N-N. This theory of bending is known as theory of simple bending.

Let

R = Radius of neutral layer N'-N'.

θ = Angle subjected at O by A'B' and C'D' produced.

y = Distance from the neutral layer.

Original length of the layer = EF = δx = NN = N'N'

From the above figure (b), N'N' = R θ

Increase in length of the EF = E'F' –EF = (R + y) θ – R θ = y θ

 Strain in the layer EF = e_{EF} = Increase in length / original length = y θ / R θ = y / R

We know strain e = stress/ Young's modulus

$$f = E\ \frac{y}{R} = \frac{E}{R}\ y$$

$$\frac{f}{y} = \frac{E}{R}$$

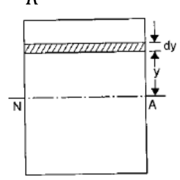
Where f is bending stress

Consider an elemental area δa at a distance y from neutral axis

The stress intensity on the elemental area

$$= f = \frac{E}{R}\ y$$

Force on the elemental area $= f\delta a = \frac{E}{R}\ y\ \delta a$



Moment of resistance offered by the elemental area = moment of thrust about Neutral Axis

$$= \frac{E}{R} y^2\ \delta a$$

Total moment of resistance offered by the beam section = $M = \frac{E}{R} \sum y^2\ \ \delta a$

But $\sum y^2\ \ \delta a$ is the moment of inertia of beam section about the neutral axis. Let this moment of inertia be I.

Therefore $$M = \frac{E}{R}\ I \qquad \frac{M}{I} = \frac{E}{R}$$

But we know $$\frac{f}{y} = \frac{E}{R}$$

Hence $$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

Where M = bending moment Nmm
I= moment of inertia of the section mm^4
f = Bending stress N/mm^2
y= distance from neutral layer to the extreme fiber of beam mm
E= Young's modulus N/mm^2
R= Radius of curvature mm

SECTION MODULUS (Z)

- It is the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis.

- Strength of a beam in bending can be estimated by its section modulus

$$Z = \frac{I}{y_{max}} \qquad mm^3$$

I = M.O.I. about neutral axis
y_{max} = Distance of the outermost layer from the neutral axis

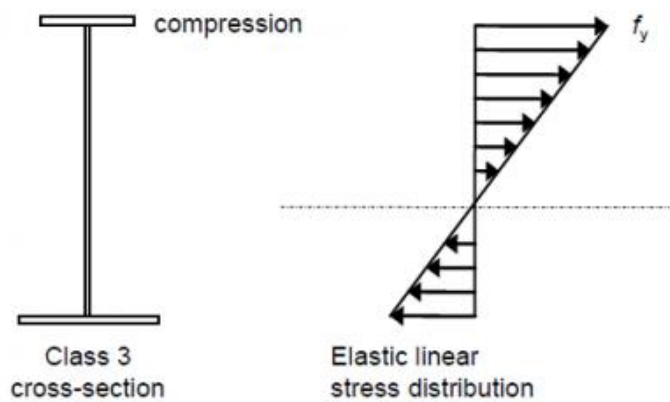
We know $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$ from this equation $\frac{M}{I} = \frac{f}{y}$

$$M = \frac{f\,I}{y} = f_{max}\,Z$$

$$M = f_{max}\,Z$$

Hence moment of resistance offered by the section is maximum when Z is maximum. Hence Z represents the strength of the section.

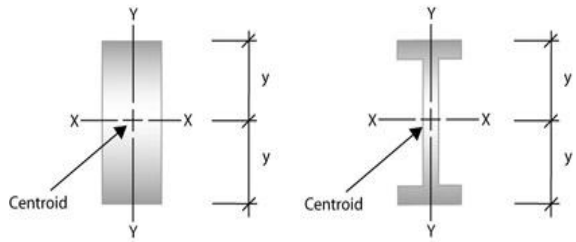
Bending Stress Distribution of I section



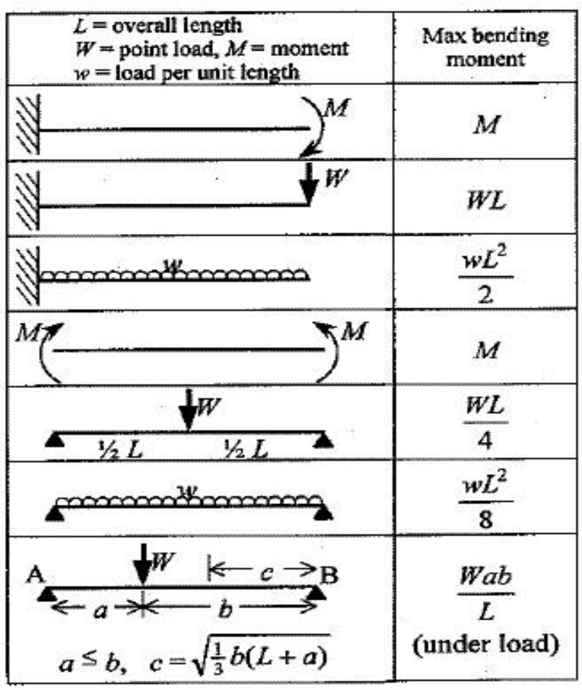
SIGNIFICANCE OF SECTION MODULUS

- The section modulus of the cross-sectional shape is of significant importance in designing beams. It is a direct measure of the **strength** of the beam.
- A beam that has a larger section modulus than another will be stronger and capable of supporting greater loads.

- It includes the idea that most of the work in bending is being done by the extreme fibres of the beam, ie the top and bottom fibres of the section.
- The distance of the fibres from top to bottom is therefore built into the calculation.
- The elastic modulus is denoted by **Z**. To calculate **Z**, the distance (y) to the extreme fibres from the centroid (or neutral axis) must be found as that is where the maximum stress could cause failure.



Maximum Bending moment for beams under different loading:



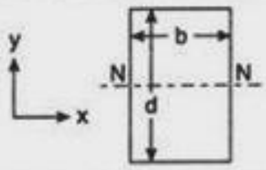
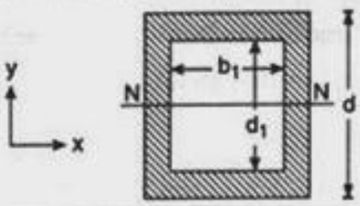
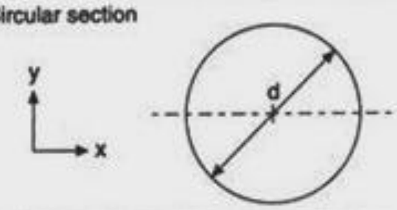
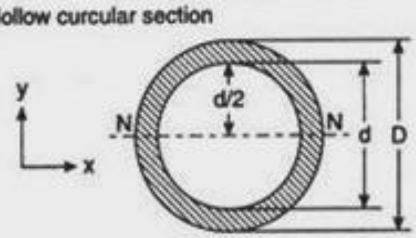
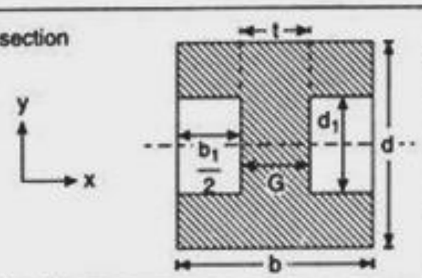
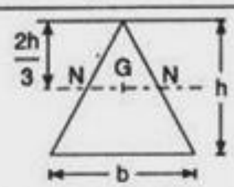
L = overall length W = point load, M = moment w = load per unit length	Max bending moment
M	M
W	WL
w	$\frac{wL^2}{2}$
M M	M
W ½ L ½ L	$\frac{WL}{4}$
w	$\frac{wL^2}{8}$
A W ← c → B ← a → ← b → $a \leq b, \quad c = \sqrt{\frac{1}{3}b(L+a)}$	$\frac{Wab}{L}$ (under load)

SIGNIFICANCE OF MOMENT OF INERTIA

- The moment of inertia of an object about a given axis describes how difficult it is to change its angular motion about that axis.
- For example, consider two discs of the same mass, one large and one small in radius. Assuming that there is uniform thickness and mass distribution, the larger radius disc requires more effort to accelerate it (i.e. change it angular motion) because its mass is effectively distributed further from its axis of rotation.

- Conversely, the smaller radius disc takes less effort to accelerate it because its mass is distributed closer to its axis of rotation. Quantitatively, the larger disc has a larger moment of inertia, whereas the smaller disc has a smaller moment of inertia.

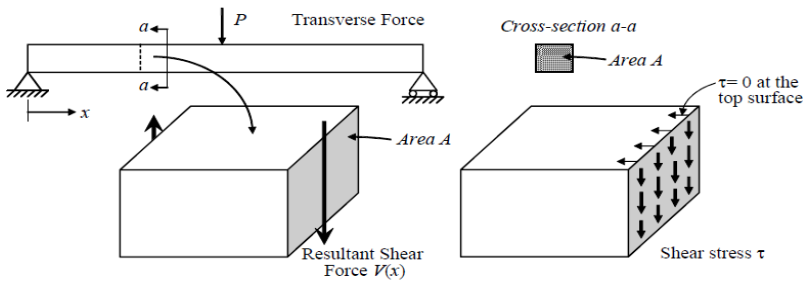
SECTION MODULUS FOR DIFFERENT SECTIONS

Type of section	Moment of Inertia	y_{max}	Section modulas (Z)
Rectangle or paralleogram 	$I_{xx} = \frac{bd^3}{12}$ $I_{yy} = \frac{db^3}{12}$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{bd^2}{6}$ $Z_{yy} = \frac{db^2}{6}$
Hollow rectangular section 	$I_{xx} = \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$ $I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{1}{6d}(bd^3 - b_1 d_1^3)$ $Z_{yy} = \frac{1}{6b}(db^3 - d_1 b_1^3)$
Circular section 	$I_{xx} = \frac{\pi}{64} d^4$ $I_{yy} = \frac{}{64} d^4$	$\frac{d}{2}$ $\frac{d}{2}$	$Z_{xx} = \frac{\pi}{32} d^3$ $Z_{yy} = \frac{}{32} d^3$
Hollow curcular section 	$I_{xx} = I_{yy} = I$ $I_{yy} = \frac{\pi}{64}(D^4 - d^4)$	$\frac{D}{2}$	$Z_{xx} = Z_{yy} = Z$ $Z = \frac{\pi}{32D}(D^4 - d^4)$
I-section 	$I_{xx} = \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$ $I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$ or $I_{xx} = \frac{1}{12}(bd^3 - (b - t) d_1^3)$	$\frac{d}{2}$ $\frac{b}{2}$	$Z_{xx} = \frac{1}{6d}(bd^3 - b_1 d_1^3)$ $Z_{yy} = \frac{1}{6b}(db^3 - d_1 b_1^3)$
Triangle 	$I_G = \frac{bh^3}{36}$	$\frac{2}{3}h$	$Z_G = \frac{bh^2}{24}$

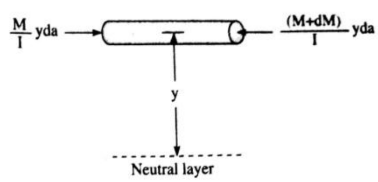
SHEAR STRESS IN BEAMS

- In addition to the pure bending case, beams are often subjected to transverse loads which generate both bending moments M(x) and shear forces F(x) along the beam.

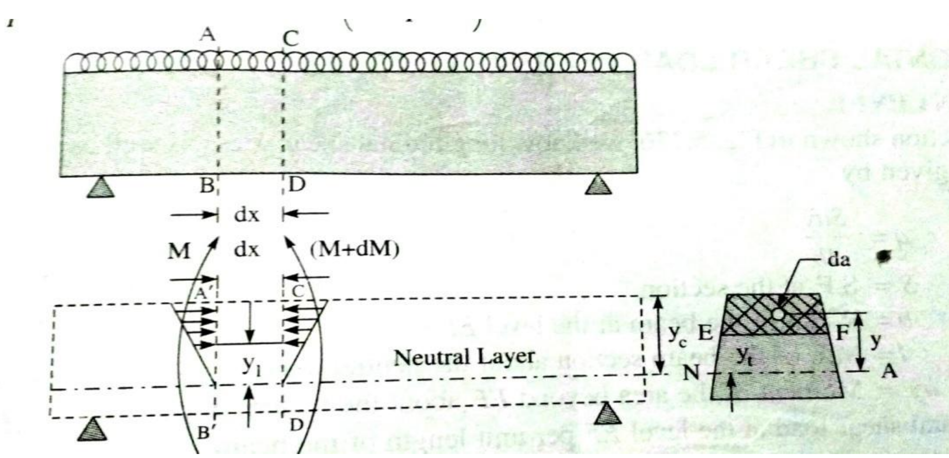
- The bending moments cause bending normal stresses to arise through the depth of the beam, and the shear forces cause transverse shear-stress distribution through the beam cross section as shown in Figure



Let any section AB the bending moment and shear force be M and F respectively. Let at another section CD distant dx from the section AB the B.M. and S.F. be (M+dM) and (F+dF) respectively



Now consider the part of the beam above the level EF and between the sections AB & CD. The part of the beam may be taken to consist of an infinite no. of elemental cylinders each of area ***da*** and length ***dx***. Consider one such elemental cylinder at a distance y from the neutral layer the intensity of the stress on the end of the elemental cylinder on the section,



$$at\ AB \quad f = \frac{M}{I} \times y$$

Similarly the intensity of stress on the end of the elemental cylinder on the section CD,

$$f + df = \frac{M + dM}{I} \times y$$

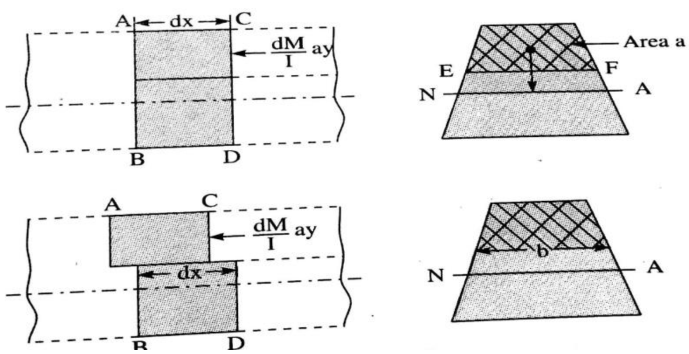
Hence the forces on the ends of the elemental cylinder are respectively,

$$f.da\ i.e., \frac{M}{I} \times y.da\ and\ (f + df)da\ i.e., \left(\frac{M + dM}{I}\right) y.da$$

Hence unbalanced force on the elemental cylinder=$\frac{dM}{I} \times y.da$

Considering all the elemental cylinders between sections AB & CD and above the level EF.

Total unbalanced force = $\sum \frac{dM}{I}.y.da = \frac{dM}{I} \sum_{y=y_t}^{y=y_c} y.da = \frac{dM}{I} \times a\bar{y}$



Where a=area of the section above the level EF and $\bar{y} = distance\ of\ the\ centroid\ of\ the\ area\ above\ the\ level\ of\ EF\ about\ the\ neutral\ axis$

Hence in order the part of the beam above the level EF and between the sections AB& CD may not fail by shear due to the unbalanced force of $\frac{dM}{I}$a$\bar{y}$ the horizontal section of the beam at the level EF must offer shear resistance. If the width of the beam at the level EF is be the intensity of the horizontal shear at the level EF

$$=q= \frac{unbalenced\ \ force}{shear\ area} = \frac{dM}{I}\frac{a\bar{y}}{dx * b} = \frac{dM}{dx}\frac{a\bar{y}}{Ib}$$

$$F=\frac{dM}{dx} = F \quad \text{hence} \ \ q = \frac{Fa\bar{y}}{Ib}$$

For the beam section shown in figure, we know longitudinal shear stress as well as vertical shear stress at this level EF is given by

$$q = \frac{Fa\bar{y}}{Ib}$$

where F= S.F at the section

b=width of the beam at the level EF

I= M.I. of the beam section about the neutral axis

$a\bar{y}$= moment of the area beyond EF about the neutral axis.

Horizontal shear load at the level EF per unit length of the beam

$$= q \times EF \times 1 = q \times b = \frac{Fa\bar{y}}{Ib} \times b = \frac{Fa\bar{y}}{I}$$

This shear load per unit length acts longitudinally at the level EF.

SHEAR STRESS IN A RECTANGULAR CROSS SECTIONED BEAM

Consider a rectangular cross sectioned beam shown in figure having width b and depth d subjected to shear force F

Consider a level *EF* at a distance y from the neutral axis. The intensity of shear stress at this level is given by

$$q = \frac{Fa\bar{y}}{Ib}$$

where $a\bar{y}$ is the moment of the area above *EF* shown shaded about the neutral axis.

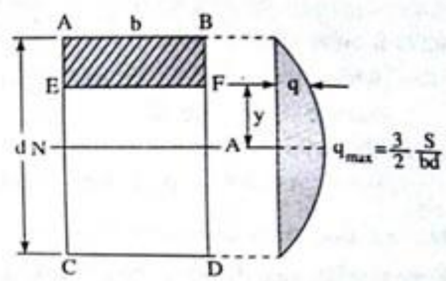


Fig. 5.127

$$\therefore \quad a\bar{y} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right) = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\therefore \quad q = \frac{F}{Ib}.\frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

But $I = \frac{bd^3}{12} \quad \therefore \quad q = \frac{12}{bd^3}.\frac{F}{b}.\frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$

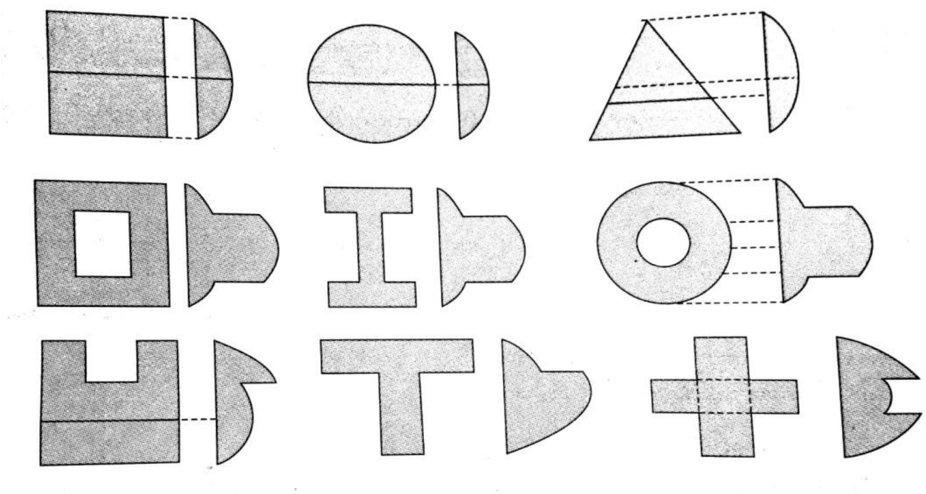
$$\therefore \quad q = \frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right)$$

At the top edge *i.e.*, at $y = \frac{d}{2}, q = 0$

At the neutral axis, *i.e.*, at $y = 0$,

$$q = \frac{6F}{bd^3}.\frac{d^2}{4} = \frac{3}{2}.\frac{F}{bd}$$

Average shear stiess $q_{avg} = \frac{F}{bd}$



Shear stress distribution for different cross sections

UNIT -IV

Deflection of beams

Objective: To make students to understand and apply concept of deflected beam and elastic curve and the students are able to determine the slope and deflection of different types of beams under applied load by using double integration,Macaulays method and moment area method.

Syllabus: Members bending into a circular arc- Equations for slope, deflection and radius of curvature Differential equation for the elastic line of a loaded beam $M = EI\, {d^2y}/{dx^2}$. Deflections in the case of cantilevers subjected to UVL and combination of loading by using double integration method. Deflections in the case of simply supported beams subjected to point load, UDL by using double integration method .Deflections in the case of simply supported beams to UVL & combination of loading by using double integration method. Deflections in the case of cantilevers subjected to point load, UDL and UVL. UVL, combine loading using Macaulay’s method. Deflections in the case of simply supported beams subjected to point load, UDL, combination of loading by using Macaulay’s method. Mohr’s theorems and their applications Moment – area method

Introduction

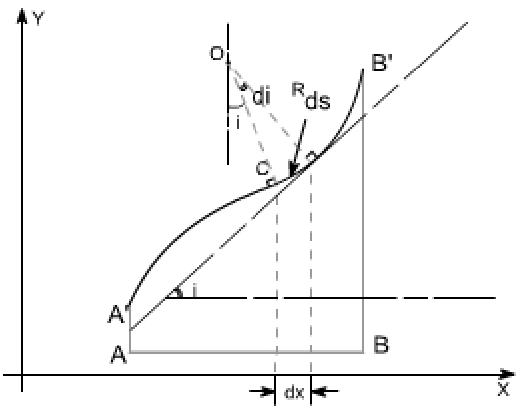
In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumptions: The following assumptions are undertaken in order to derive a differential Equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflects to a position A'B' under load or in fact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Further, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.
To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection. Further, let us consider element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or} \quad \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force, bending moment and deflection of the beam may be obtained as differentiating the equation as derived

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx} = F$$

Thus,

$$F = EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods. The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct/Double integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

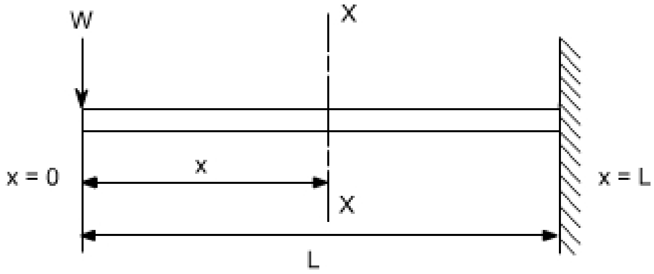
Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Example: let us consider few illustrative examples to have a familiarity with the direct integration method

Case 1: Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force and bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI} dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI} dx + \int A \, dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)
at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

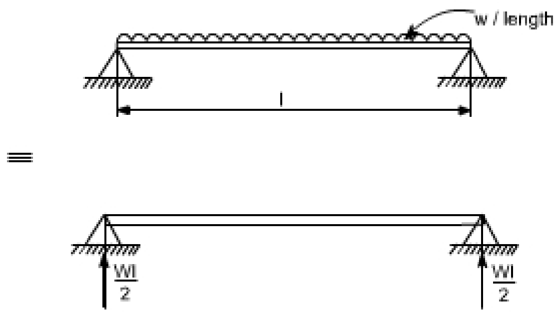
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

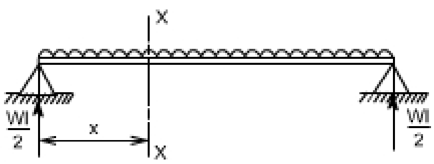
$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(Slope)_{max}m = +\frac{WL^2}{2EI}}$$

Case 2: Simply Supported beam with uniformly distributed Loads:- In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x meter from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{- - - - - (1)}$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary Conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Further

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then} \quad y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

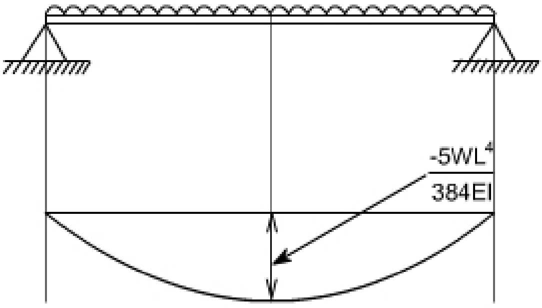
Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) The value of maximum deflection would be at the centre i.e. at x = L/2. The final equation which is governs the deflection of the loaded beam in this case is

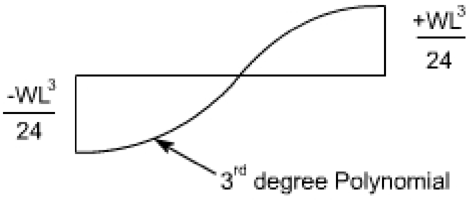
$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.



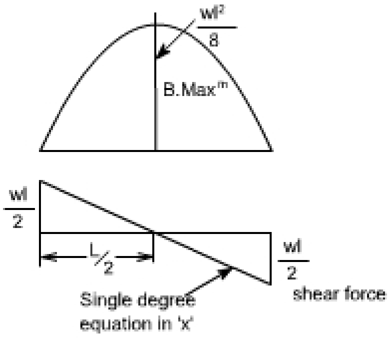
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$

Shear Force

Shear force is obtained by taking third derivative

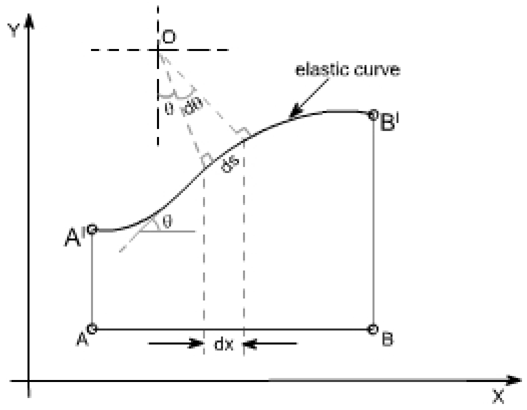
$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

THE AREA-MOMENT /MOHRS MOMENT-AREA METHODS:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple. Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that dq is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.

We can assume,
ds = dx [since the curvature is small]
hence, R dq = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

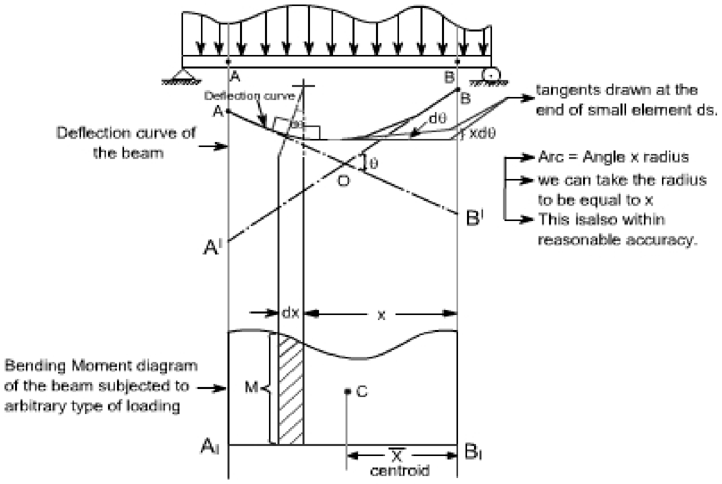
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle ,slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$,hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \quad ----- (1)$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and A1B1is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Further, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are further understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be dq. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI. From the above relationship the total angle q between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or } \theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x dq [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle q]

Hence the total distance B'B becomes $\delta = \int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of dq = M dx / EI as derived earlier

$$\delta = \int_A^B x \frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$

[This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI. Therefore, we are in a position to state the above conclusion in the form of theorem as
follows:

Theorem II:

Deflection of point 'B' relative to point A

$$= \frac{1}{EI} \times \left\{ \begin{array}{l} \text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram} \end{array} \right\}$$

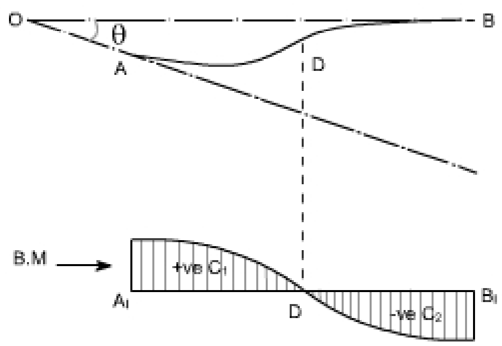
Further, the first moment of area, according to the definition of centroid may be written

as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $$\boxed{\delta_A = \frac{1}{EI} A\bar{X}}$$

Therefore, the first moment of area may be obtained simply as a product of the total area of the B.M diagram between the points A and B multiplied by the distance to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroid C1and C2. Then to find an angle q between the tangents at the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

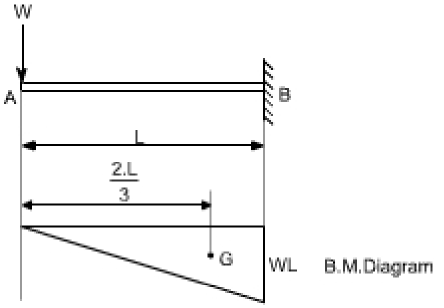
And similarly for the deflection of B away from the tangent at A becomes

$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Example 1:

1. A cantilever is subjected to a concentrated load at the free end. It is required to find out the deflection at the free end.

For a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - Moment theorem

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between the points A and B}]$$

$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$

$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

NOTE: In this case the point B is at zero slope

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$

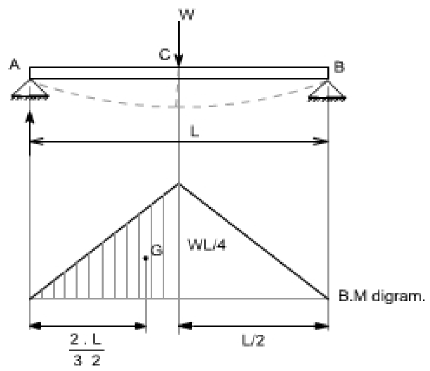
$$= \frac{1}{EI}[A\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at A} = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

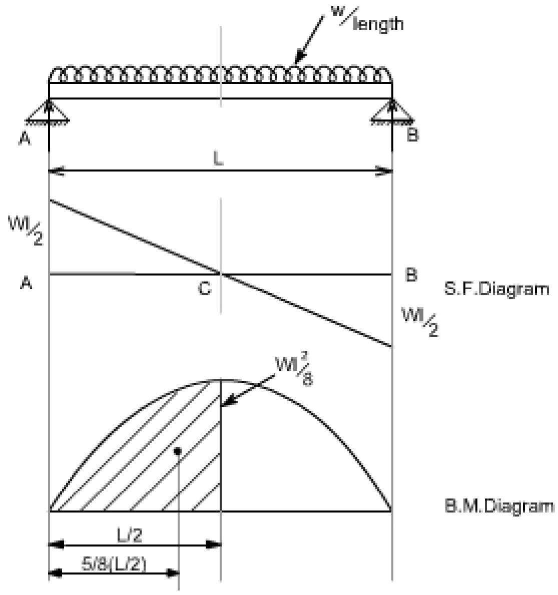
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with a intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam; the value of maximum B.M is equal to $Wl^2 / 8$



So by area moment method

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}\left[\text{Area of B.M diagram between point A and C}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}\left[A\,\bar{y}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

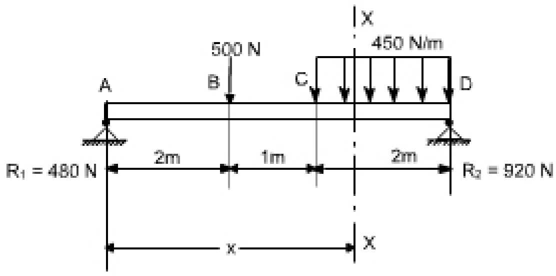
MACAULAY'S METHOD

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

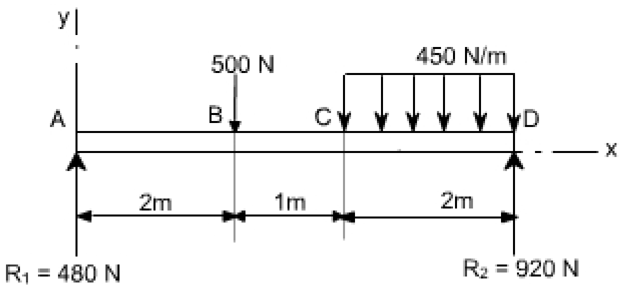
For example consider the beam shown in fig below:

Let us write the general moment equation using the definition M = (Σ M)L, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows



$$M_{AB} = 480\,x \text{ N.m}$$
$$M_{BC} = \left[480\,x - 500(x-2)\right] \text{N.m}$$
$$M_{CD} = \left[480\,x - 500(x-2) - \frac{450}{2}(x-3)^2\right] \text{N.m}$$

It may be observed that the equation for MCD will also be valid for both MAB and MBC provided that the terms (x - 2) and (x - 3)2are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and (x - 3)2 are nonexistent for values of x for which the terms in parentheses are negative

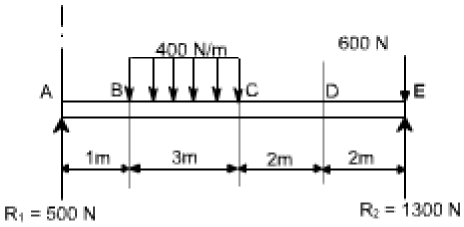


As an clear indication of these restrictions, one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

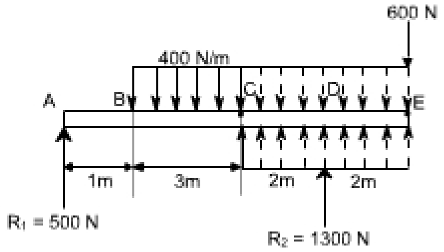
$$M = \left(480x - 500(x-2) - \frac{450}{2}(x-3)^2\right) \text{N.m}$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:



(a)



$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$

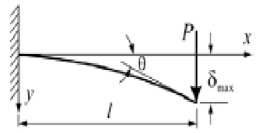
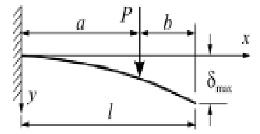
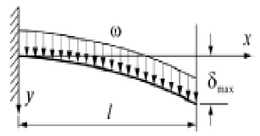
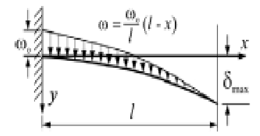
It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratory just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

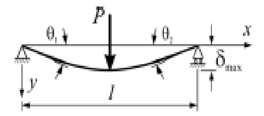
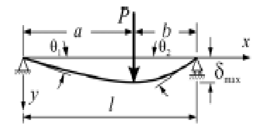
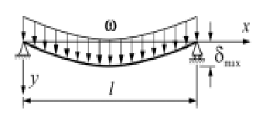
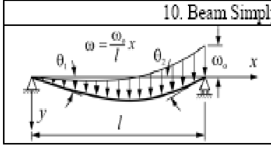
Procedure to solve the problems

(i). after writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.
(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.
(iii) integrating on both sides and apply the boundary conditiond and finding thr ingration constants.
(iv) Substitute the integration constants and finding the valuesn of slope and deflection at required locations.

IMPORTANT FORMULAS

BEAM DEFLECTION FORMULAE

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load P at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load P at any point			
	$\theta = \frac{Pa^2}{2EI}$	$y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam – Uniformly distributed load ω (N/m)			
	$\theta = \frac{\omega l^3}{6EI}$	$y = \frac{\omega x^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{\omega l^4}{8EI}$
4. Cantilever Beam – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta = \frac{\omega_o l^3}{24EI}$	$y = \frac{\omega_o x^2}{120lEI}(10l^3 - 10l^2x + 5lx^2 - x^3)$	$\delta_{max} = \frac{\omega_o l^4}{30EI}$

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \frac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \frac{Pab(2l - b)}{6lEI}$	$y = \frac{Pbx}{6lEI}(l^2 - x^2 - b^2)$ for $0 < x < a$ $y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + (l^2 - b^2)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \frac{Pb(l^2 - b^2)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{(l^2 - b^2)/3}$ $\delta = \frac{Pb}{48EI}(3l^2 - 4b^2)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \frac{\omega l^3}{24EI}$	$y = \frac{\omega x}{24EI}(l^3 - 2lx^2 + x^3)$	$\delta_{max} = \frac{5\omega l^4}{384EI}$
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \frac{7\omega_o l^3}{360EI}$ $\theta_2 = \frac{\omega_o l^3}{45EI}$	$y = \frac{\omega_o x}{360lEI}(7l^4 - 10l^2x^2 + 3x^4)$	$\delta_{max} = 0.00652\frac{\omega_o l^4}{EI}$ at $x = 0.519l$ $\delta = 0.00651\frac{\omega_o l^4}{EI}$ at the center

UNIT-VI :THIN CYLINDERS AND THICK CYLINDERS

- **Objective:** To make students to understand and apply concept of thin, thick cylinders and Lame's equation.

Syllabus

Thin cylinders: Thin seamless cylindrical shells, Derivation of the formula for hoop stress and longitudinal stress, Hoop strain and longitudinal strain Volumetric strain, Riveted boiler shells Thin spherical shells.

Thick cylinders: Introduction thick cylinders, Lames equation, Stresses in Cylinders subjected to inside and outside pressures compound Cylinders.

THIN CYLINDERS

Introduction

- Cylindrical or spherical pressure vessels (e.g., hydraulic cylinders, gun barrels, pipes, boilers and tanks) are commonly used in industry to carry both liquids and gases under pressure.
- When the pressure vessel is exposed to this pressure, the material comprising the vessel is subjected to pressure loading, and hence stresses, from all directions.
- The normal stresses resulting from this pressure are functions of the radius of the element under consideration, the shape of the pressure vessel (i.e., open ended cylinder, closed end cylinder, or sphere) as well as the applied pressure.
- If the thickness of the wall of the cylindrical vessel is less than 1/15 to 1/20 of its internal diameter, the cylindrical vessel is known as thin cylinder.
- The second method is based on elasticity solution and is always applicable regardless of the d/t ratio and can be referred to as the solution for "thick wall" pressure vessels.

Thin-Walled Pressure Vessels

Several assumptions are made in this method.

1. thickness of the wall of the cylindrical vessel is less than 1/15 to 1/20 of its internal diameter
2. Plane sections remain plane
3. t being uniform and constant
4. The applied pressure, p, is the gage pressure (note that p is the difference between the absolute pressure and the atmospheric pressure)
5. Material is linear-elastic, isotropic and homogeneous.
6. Stress distributions throughout the wall thickness will not vary
7. Element of interest is remote from the end of the cylinder and other geometric discontinuities.
8. Working fluid has negligible weight

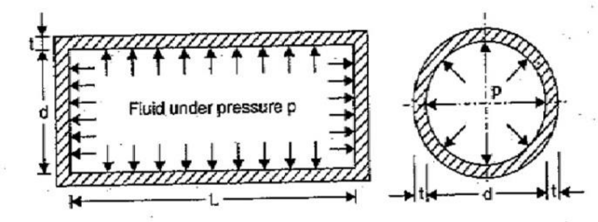
Thin cylindrical vessel subjected to internal pressure

let d= internal diameter of

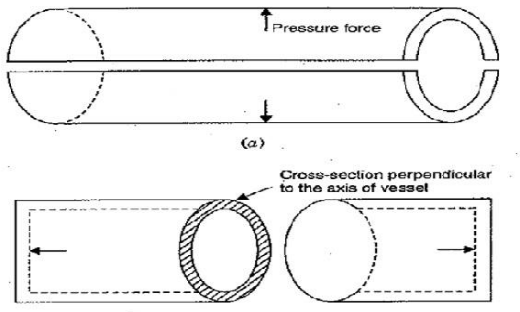
t= thickness of the wall of the cylinder

p= internal pressure of the fluid

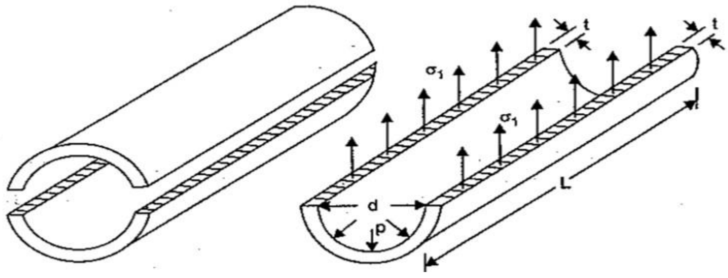
L= length of the cylinder



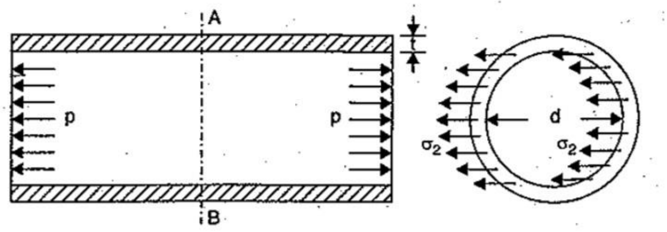
on account of internal pressure p, the cylindrical vessel may fail by splitting up in any one of the two ways as shown in figures



The forces due to pressure of the fluid acting vertically upward and downwards on the cylinder tend to burst the cylinder as shown in figures.



The force due to pressures of the fluid, acting at the ends of the thin cylinder, tend to burst the thin cylinder as show in figure



Stresses in a thin cylindrical vessel subjected to internal pressure:

When a thin cylindrical vessel is subjected to internal fluid pressure, the stresses in the wall of cylinder on the on the cross section along the axis and on the cross section perpendicular to the axis are set up. These stresses are tensile and are known as

1. Circumferential stress (or hoop stress)
2. Longitudinal stress

The name of the stress is given according to the direction in which the stress is acting.

- The stress acting along the circumference of the cylinder is called circumferential stress where as the stress acting along the length of the cylinder is known as longitudinal stress.

Expression for circumferential stress (or Hoop stress):

Consider a cylindrical vessel subjected to an internal fluid pressure. The circumferential stress will be set up in the material of the cylinder. If the bursting of the cylinder takes place as shown in the figure
The expression for Hoop stress or circumferential stress (σ_1) is obtained as given below

Let p= internal fluid pressure
d= internal diameter of the cylinder
t= thickness of the wall of the cylinder
σ_1= circumferential or hoop stress

The bursting will take place if the force due to fluid pressure is more than the resisting force due to circumferential stress set up in the material. In the limiting case the two forces should be equal

Force due to fluid pressure = p x area on which p acting

= p x (d x L) (i)
(since p is acting on projected area d x L)

Force due to circumferential stress
= σ_1 x area on which σ_h is acting
= σ_1 x (L x t + L x t)
= σ_1 x 2Lt = 2 σ_1 x L x t (ii)

equating (i) and (ii) we get

p x (d x L) = 2 σ_1 x L x t

$$\sigma_1 = \frac{pd}{2t}$$ this stress is tensile in nature

Expression for longitudinal stress

Consider a thin cylindrical vessel subjected to an internal fluid pressure. The longitudinal stress will be set up in the material of the cylinder, if the bursting takes place along the section AB as shown in figure

The longitudinal stress (σ_2) developed in the material is obtained as

Let p = internal pressure of the fluid stored in the cylinder
d= internal diameter of the cylinder
t= thickness of the cylinder
σ_2 = longitudinal stress in the material

The bursting will take place if the force due to the fluid pressure acting on the ends of the is more than the resisting force due to longitudinal stress (σ_2) developed in the material as shown in the figure

in the limiting case both force should be equal

force due to fluid pressure =p x area on which p is acting

$$= p \text{ x } \frac{\pi}{4} d^2$$

Resisting force = σ_2 x area on which σ_L is acting

= σ_2 x πd x t

hence in the limiting case

force due to fluid pressure = Resisting force

$$\sigma_2 \text{ x } \pi d \text{ x } t = p \text{ x } \frac{\pi}{4} d^2$$

$$\sigma_2 = \frac{pd}{4t}$$ This stress is also tensile in nature

From the longitudinal stress and hoop stress formulae we can justify, longitudinal stress is half the circumferential stress.

$$\sigma_2 = \frac{\sigma_1}{2}$$

This also means that circumferential stress is two times the longitudinal stress. Hence in the material of the cylinder the permissible stress should be less than the longitudinal stress.

Maximum shear stress: at any point on the material of the cylinder shell there are two principal stresses namely circumferential stress of magnitude $\sigma_1 = \frac{pd}{2t}$ acting circumferentially and longitudinal stress of magnitude $\sigma_2 = \frac{pd}{4t}$ acting parallel to the axis of the shell. These two stresses are tensile and perpendicular to each other.

Maximum shear stress $$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2} = \frac{pd}{8t}$$

Efficiency of the joint:

- The cylinders such as boilers are having two types of joints namely longitudinal and circumferential joint.
- In case of making joints holes are made in the material of the shell for riveting.
- Due to the holes the area offering resistance decreases. This results in stress developed in the material of the shell will be more.

If the efficiency of a longitudinal joint and circumferential joint are given then circumferential and longitudinal stresses are obtained as

Let η_L = efficiency of longitudinal joint

η_h = efficiency of circumferential joint

then the circumferential stress σ_1 is given by

$$\sigma_1 = \frac{pd}{\eta_L\, 2t}$$

and the longitudinal stress σ_1 is given by

$$\sigma_2 = \frac{pd}{\eta_h\, 4t}$$

- In longitudinal joint the circumferential stress is developed where as in circumferential joint longitudinal stress is developed

Dimensional changes of a thin cylindrical shell due to internal pressure:

- Due to internal fluid pressure in a cylindrical shell dimensional changes will occur. These dimensional changes are change in longitudinal, circumferential and volumetric changes.

Let p = internal fluid pressure
L = Length of the cylinder
d = Diameter of the cylinder
t = Thickness of the cylinder
E = Young's modulus of the material
σ_1= Hoop stress in material
σ_2= Longitudinal stress in the material
μ = poison's ratio
δd = change in diameter due to stress developed in the material
δL= change in length
δV = change in volume

the value of $\sigma_1$and $\sigma_2$are given by $\sigma_1 = \frac{pd}{2t}$ and $\sigma_2 = \frac{pd}{4t}$

let e_1 = Circumferential strain
e_2 = Longitudinal strain

then the **circumferential strain,**

$$e_1 = \frac{\sigma_1}{\boldsymbol{E}} - \frac{\mu\sigma_2}{\boldsymbol{E}}$$

$$= \frac{pd}{2tE} - \frac{\mu pd}{4tE}$$

$$e_1 = \frac{pd}{2tE}\left[1 - \frac{\mu}{2}\right]$$

and longitudinal strain,

$$e_2 = \frac{\sigma_2}{\boldsymbol{E}} - \frac{\mu\sigma_1}{\boldsymbol{E}}$$

$$= \frac{pd}{4tE} - \frac{\mu pd}{2tE}$$

$$e_2 = \frac{pd}{2tE}\left[\frac{1}{2} - \mu\right]$$

but circumferential strain for cylinder is also given as

$$e_1 = \frac{change\ in\ circumference\ due\ to\ pressure}{original\ circumference}$$

$$e_1 = \frac{final\ circumference - original\ circumference}{original\ circumference}$$

$$e_1 = \frac{\pi(d+\delta d) - \pi d}{\pi d}$$

$$= \frac{\delta d}{d} \qquad or = \left[\frac{change\ in\ diameter}{original\ diameter}\right]$$

Equating the two values of e_1

$$\frac{\delta d}{d} = \frac{pd}{2tE}\left[1 - \frac{\mu}{2}\right]$$

$$\delta d = \frac{pd^2}{2tE}\left[1 - \frac{\mu}{2}\right]$$

Similarly longitudinal strain is also given as

$$e_2 = \frac{change\ in\ length\ due\ to\ pressure}{original\ length}$$

$$= \frac{\delta L}{L}$$

Equating two values of e_2

$$\frac{\delta L}{L} = \frac{pd}{2tE}\left[\frac{1}{2} - \mu\right]$$

$$\delta L = \frac{pdL}{2tE}\left[\frac{1}{2} - \mu\right]$$

Volumetric strain: it is defined as change in volume to the original volume of the cylinder

Volumetric strain $= \frac{\delta V}{V}$

Change in volume (δV) = Final volume – original volume

Original volume (v) = Area of cylindrical shell x Length

$= \frac{\pi}{4} d^2 \text{ x } L$

Final volume = (Final area of cross section) x final length

$= \frac{\pi}{4}\ [d + \delta d]^2 \text{x} [L + \delta L\ ]$

Expanding the above equation and neglecting the smaller quantities we get final volume as

Final volume $= \frac{\pi}{4}\ [d^2 L + 2d\ L\delta d +\ \delta L\ d^2] - \frac{\pi}{4}\ d^2 \text{ x } L$

$= \frac{\pi}{4}\ [2d\ L\delta d +\ \delta L\ d^2]$

Volumetric strain $\frac{\delta V}{V} = \frac{\frac{\pi}{4}[2d\ L\delta d + \delta L\ d^2]}{\frac{\pi}{4} d^2 \text{ x } L}$

$= \frac{2\delta d}{d} + \frac{\delta L}{L}$

$= 2\ e_1 + e_2$

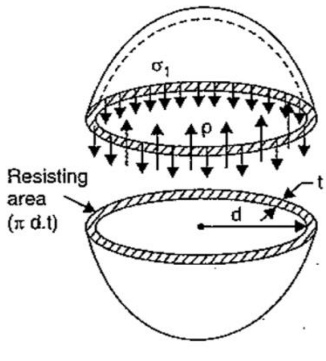
$= 2 \text{ x} \frac{pd}{2tE}\left[1 - \frac{\mu}{2}\right] + \frac{pd}{2tE}\left[\frac{1}{2} - \mu\right]$

Simplifying the above equation we get

$$\frac{\delta V}{V} = \frac{pd}{2tE}\left[\frac{5}{2} - 2\mu\right]$$

Also change in volume (δV) = V $(2e_1+e_2)$

Thin spherical shells:



Above figure shows thin spherical shell of internal diameter ‘d’ and thickness ‘t’ and subjected to an internal fluid pressure ‘p’. The fluid inside the shell has a tendency to split the shell into two hemispheres along x-x axis.

The force (P) which has a tendency to split the shell

$= p \text{ x } \frac{\pi}{4} d^2$

The area resisting this force $= \pi.d.t$

Hoop or circumferential stress $\sigma_1 = \frac{force\ P}{area\ resisting\ the\ force\ P}$

$$= \frac{p \text{ x} \frac{\pi}{4} d^2}{\pi.d.t} = \frac{pd}{4t}$$

the stress σ_1 is tensile in nature the fluid inside the shell is also having the tendency to split into two hemispheres along y-y axis. Then it can be shown the tensile hoop stress will also be equal to $\frac{pd}{4t}$. Let this stress is σ_2.

$$\sigma_2 = \frac{pd}{4t}$$

Change in dimensions of spherical shell due to an internal pressure:

There is no shear stress acts on spherical shells

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\frac{pd}{4t} - \frac{pd}{4t}}{2} = 0$$

Strain in any one direction is given by,

$$e = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E}$$
$$= \frac{\sigma_1}{E} - \frac{\mu\sigma_1}{E}$$
$$= \frac{\sigma_1}{E}(1-\mu)$$
$$= \frac{pd}{4tE}[1-\mu]$$

We know that strain in any direction is also

$$= \frac{\delta d}{d}$$

then $$\frac{\delta d}{d} = \frac{pd}{4tE}[1-\mu]$$

Volumetric strain ($\frac{\delta V}{V}$):

The ratio of change in volume to the original volume is known as volumetric strain.

If V= original volume and δV = change in volume, then $\frac{\delta V}{V}$ = volumetric strain

Let V= original volume

$$= \frac{\pi}{6} d^3$$

taking the difference of the above equation, we get

$$\delta V = \frac{\pi}{6} \text{ x } 3\, d^2 \text{ x } \delta(d)$$
$$\frac{\delta V}{V} = \frac{\frac{\pi}{6} \text{ x } 3\, d^2 \text{ x } \delta(d)}{\frac{\pi}{6} d^3}$$
$$= 3\frac{\delta(d)}{d}$$

We know that $$\frac{\delta d}{d} = \frac{pd}{4tE}[1-\mu]$$

Then volumetric strain $$\frac{\delta V}{V} = \frac{3pd}{4tE}[1-\mu]$$

THICK CYLINDERS

The problem of determination of stresses in thick cylinders was first attempted more than 160 years ago by a French mathematician Lame in 1833. His solution very logically assumed that a thick cylinder to consist of series of thin cylinders such that each exerts pressure on the other.

This will essentially focus attention on three stress components at any point these stress components are:

• 1) Stress along the circumferential direction, called hoop or tangential stress.

• 2) Radial stress which is stress similar to the pressure on free internal or external surface. (This stress will also vary in the radial direction & not with 'Θ' as in tangential stress case.)

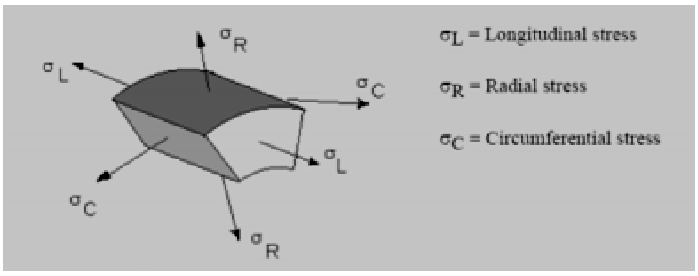
• 3) Longitudinal stress in the direction the axis of the cylinder. This stress is perpendicular to the plane of the paper. So the longitudinal stress will remain same /constant for any section of the thick cylinder.

This will be associated with the assumption that any section of thick cylinder will remain plane before & after the application of pressure.

• This assumption will mean that the strain along the axis or length remain constant.

• Thick cylinders also have the external pressure, not only the internal pressure.

Lame's Theory

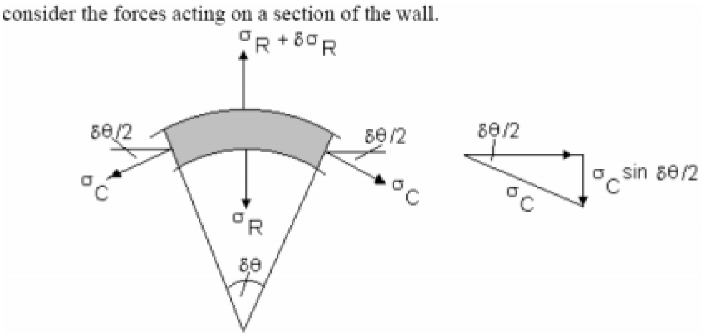


$$\epsilon_L = \frac{1}{E} * (\sigma_L - \vartheta(\sigma_R + \sigma_C)$$

$$\epsilon_c = \frac{1}{E} * (\sigma_c - \vartheta(\sigma_R + \sigma_L)$$

$$\epsilon_R = \frac{1}{E} * (\sigma_R - \vartheta(\sigma_L + \sigma_C)$$

consider the forces acting on a section of the wall.



$$(\sigma_r + d\sigma_r)(r + dr)d\theta - \sigma_r * rd\theta = 2\sigma_H * dr * sin\frac{d\theta}{2}$$

For small angles: $sin\frac{d\theta}{2} = \frac{d\theta}{2}$

Therefore, neglecting second-order small quantities,

$$\text{r}\ d\sigma_r + \sigma_r dr = \sigma_H dr$$

$$\sigma_r + \frac{rd\sigma_r}{dr} = \sigma_H$$

$$\sigma_H - \sigma_r = \frac{rd\sigma_r}{dr}$$

Assuming now that plane sections remain plane, ie. The longitudinal strain is constant across the wall of the cylinder

$\in_L = \frac{1}{E} * (\sigma_L - \vartheta(\sigma_R + \sigma_H)$= constant ……………….1

It is also assumed that the longitudinal stress is constant across the cylinder walls at points remote from the ends.

$\sigma_R + \sigma_H = Constant = 2A$…………………………2

from eq 1 and 2

2A-$\sigma_R - \sigma_{R}$= $\frac{rd\sigma_r}{dr}$ muliply with r and rewrite equation

$$\frac{d}{dr}(\sigma_R r^2 - Ar^2) = 0$$

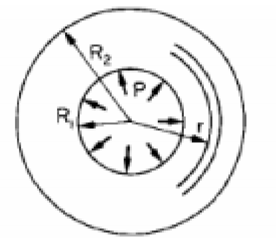
Integrating on both sides then $\sigma_R = A - B/r^2$

And $\sigma_H = A + B/r^2$

The above equations yield the radial and hoop stresses at any radius r in terms of constants A and B. For any pressure condition there will always be two known conditions of stress

Thick cylinder - internal pressure only

• Consider now the thick cylinder shown in Fig. subjected to an internal pressure P, the external pressure being zero. • The two known conditions of stress which enable the Lame constants A and B to be determined are:

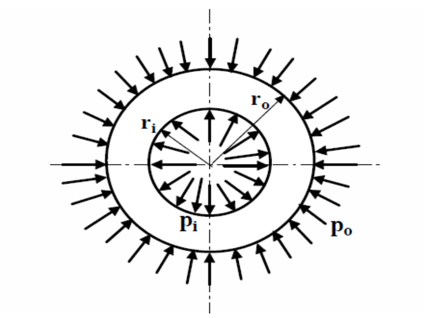


At r=R1, $\sigma_R = -P$ and at r=R2, $\sigma_R = 0$

Substitute the boundary conditions in Lames equation then

Hoop stress $\sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)} * (1 + R_2^2/r^2)$

similarly, $\sigma_r = -p_i$



A thick cylinder with both external and internal pressure

Boundary conditions for a thick cylinder with internal and external pressures p_i and p_o respectively are:

at $r = r_i \quad \sigma_r = -p_i$

and at $r = r_o \quad \sigma_r = -p_o$

The negative signs appear due to the compressive nature of the pressures. This gives

$$c_1 = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \qquad c_2 = \frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2}$$

The radial stress σ_r and circumferential stress σ_θ are now given by

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2} \frac{1}{r^2}$$

$$\sigma_\theta = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2} \frac{1}{r^2}$$

It is important to remember that if σ_θ works out to be positive, it is tensile and if it is negative, it is compressive whereas σ_r is always compressive irrespective of its sign.

Stress distributions for different conditions may be obtained by simply substituting the relevant values in corresponding equation

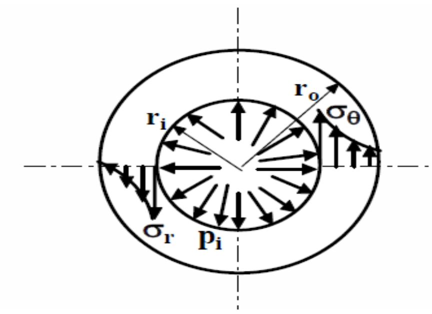
"(here the constants c_1,c_2 represents A,B)"

And hoop stress is $\sigma_\Theta = \sigma_h$

For example, if $p_o = 0$ i.e. there is no external pressure the radial and circumferential stress reduce to

$$\sigma_r = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(-\frac{r_o^2}{r^2} + 1\right)$$

$$\sigma_\theta = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(\frac{r_o^2}{r^2} + 1\right)$$



Radial and circumferential stress distribution within the cylinder wall when only internal pressure acts.

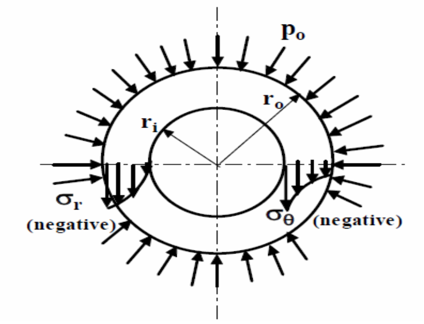
It may be noted that $\sigma_r + \sigma_\theta$ = constant and hence the deformation in z-direction is uniform. This means that the cross-section perpendicular to the cylinder axis remains plane. Hence the deformation in an element cut out by two adjacent cross-sections does not interfere with the adjacent element

If $p_i = 0$ i.e. there is no internal pressure the stresses σ_r and σ_θ reduce to

$$\sigma_r = \frac{p_o r_o^2}{r_o^2 - r_i^2}\left(\frac{r_i^2}{r^2} - 1\right)$$

$$\sigma_\theta = -\frac{p_o r_o^2}{r_o^2 - r_i^2}\left(\frac{r_i^2}{r^2} + 1\right)$$

The stress distributions are shown as



Distribution of radial and circumferential stresses within the cylinder wall when only external pressure acts.

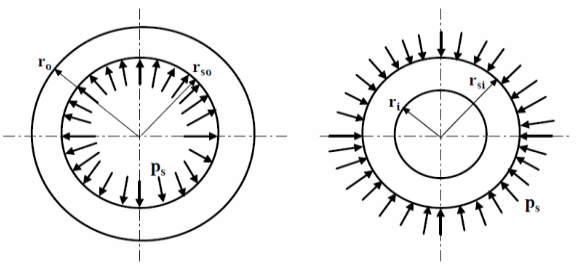
In order to make thick-walled cylinders that resist elastically large internal pressure and make effective use of material at the outer portion of the cylinder the following methods of pre-stressing are used:

1. Shrinking a hollow cylinder over the main cylinder.
2. Multilayered or laminated cylinders.
3. Autofrettage or self hooping.

1. Composite cylinders

An outer cylinder (jacket) with the internal diameter slightly smaller than the outer diameter of the main cylinder is heated and fitted onto the main cylinder. When the assembly cools down to room temperature a composite cylinder is obtained. In this process the main cylinder is subjected to an external pressure leading to a compressive radial stress at the interface. The outer cylinder or the jacket is subjected to an internal pressure leading to a tensile circumferential stress at the inner wall. Under this condition as the internal pressure increases the compression in the inner cylinder is first released and then only the cylinder begins to act in tension. Gun barrels are normally pre-stressed by hooping since very large internal pressures are generated. Here the main problem is to determine the contact pressure p_s. At the contact surface the outer radius r_{si} of the inner cylinder is slightly larger than the inside diameter r_{so} of the outer cylinder. However for stress calculations we assume that $r_{so} \simeq r_{si} = r_s$ (say).

The inner and outer cylinders are shown as



Jacket or outer cylinder

Inner cylinder

Dimensions and the pressures at the contact surface of the internal and outer cylinders

For the <u>outer cylinder</u> the radial and circumferential stresses at the contact surface may be given by

$$\sigma_r\big|_{r=r_s} = \frac{p_s r_s^2}{r_o^2 - r_s^2}\left(1 - \frac{r_o^2}{r_s^2}\right) = -p_s$$

$$\sigma_\theta\big|_{r=r_s} = \frac{p_s r_s^2}{r_o^2 - r_s^2}\left(1 + \frac{r_o^2}{r_s^2}\right)$$

In order to find the radial displacements of the cylinder walls at the contact we consider that

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

This gives the radial displacement of the inner wall of the outer cylinder as

$$u_{r1} = \frac{p_s r_s}{E}\left[\frac{r_o^2 + r_s^2}{r_o^2 - r_s^2} + \nu\right]$$

Similarly for the <u>inner cylinder</u> the radial and circumferential stresses at the outer wall can be given by

$$\sigma_r\big|_{r=r_s} = -\,p_s$$

$$\sigma_\theta\big|_{r=r_s} = -p_s\frac{r_s^2 + r_i^2}{r_s^2 - r_i^2}$$

And following the above procedure the radial displacement of the contact surface of the inner cylinder is given by

$$u_{r2} = -\frac{p_s r_s}{E}\left[\frac{r_s^2 + r_i^2}{r_s^2 - r_i^2} - \nu\right]$$

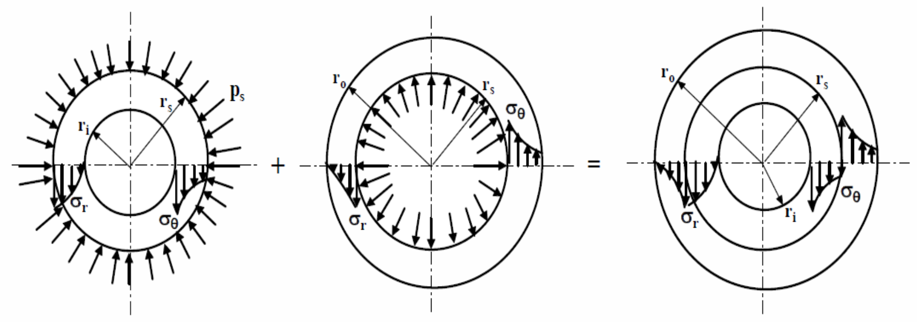
The total interference δ at the contact is therefore given by

$$\delta = \frac{p_s r_s}{E}\left[\frac{r_o^2 + r_s^2}{r_o^2 - r_s^2} + \frac{r_s^2 + r_i^2}{r_s^2 - r_i^2}\right].$$

This gives the contact pressure in terms of the known variables as follows:

$$p_s = \frac{E\delta}{r_s\left[\frac{r_o^2 + r_s^2}{r_o^2 - r_s^2} + \frac{r_s^2 + r_i^2}{r_s^2 - r_i^2}\right]}$$

The combined stress distribution in a shrink fit composite cylinder is made up of stress distribution in the inner and outer cylinders and this is shown as



Combined stress distribution in a composite cylinder

Residual circumferential stress is maximum at r = r_i for the inner cylinder and is given by

$$\sigma_{\theta(max)}\Big|_{r=r_i} = -\frac{2p_s r_s^2}{r_s^2 - r_i^2}$$

Residual circumferential stress is maximum at r = r_s for the outer cylinder and is given by

$$\sigma_{\theta(max)}\Big|_{r=r_s} = p_s \frac{r_o^2 + r_s^2}{r_o^2 - r_s^2}$$

Stresses due to fluid pressure must be superimposed on this to find the complete stress distribution.

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I) Objective/fill in the blanks Questions

1. Thickness of cylinder when a cylindrical shell subjected to internal fluid pressure ‘p’, diameter ‘d’ having maximum hoop stress in the material σ_1 is __________________
2. Thickness of cylinder when a cylindrical shell subjected to internal fluid pressure ‘p’, diameter ‘d’ having maximum longitudinal stress in the material σ_2 is __________________
3. Shear stress in a thin spherical pressure vessel is ________________
4. Hoop stress in thin spherical pressure vessel having diameter ‘d’ and thickness ‘t’ subjected to an internal fluid pressure ‘p’ is________________
5. In a thick walled cylinder subjected to an internal fluid pressure, maximum hoop stress occurs at
   a. Outer wall b. inner wall c. midpoint of thickness
6. The maximum stress in a thick cylinder is
   a. Radial stress b. hoop stress c. longitudinal stress
7. The use of compound tubes subjected to internal pressure are made to
   a. Even out the stress
   b. increase the thickness
   c. increase the diameter of the tube
   d. increase the strength

II Descriptive questions

1. Distinguish between thin cylinder and thick cylinder
2. What are the stresses developed in a thin cylinder subjected to internal pressure
3. What are the expressions for longitudinal and circumferential stresses of a thin cylinder subjected to internal pressure?
4. Find the change in length and change in diameter of a cylindrical thin shell subjected to internal fluid pressure
5. What is a compound cylinder what are its applications
6. Define volumetric strain in thin cylinder subjected to internal pressure and what is its expression
7. What are the stresses developed in a thick cylinder subjected to internal pressure
8. What are the assumptions made in the Lame’s theorem
9. what are the techniques used to increase pressure carrying capacity of a cylinder.
10. what are the expressions for dimensional changes in case of a cylinder subjected to internal pressure.

A. Question testing the ability of students in applying the concepts.

1. Cylindrical pipe of diameter 2.0 m and thickness 2.0 cm is subjected to an internal fluid pressure of 1.5 N/mm^2. Determine
   i) Hoop stress
   ii) Longitudinal stress developed in the material
2. A thick cylinder of 200 mm outside diameter and 140 mm inside diameter is subjected to internal pressure of 40 Mpa and external pressure of 24 Mpa. Determine the maximum shear stress in the material of the cylinder at the inside diameter
3. A 6 m long thin cylindrical shell is 800 mm in diameter and 10 mm thick .It is subjected to an internal pressure of 4 Mpa. Determine the change in diameter, change in length and change in volume of the shell. E=200 Gpa and poisson's ratio 0.3.
4. A 6 m long thin cylindrical shell is 800mm in diameter and 10mm thick. Is subjected to an internal pressure of 4 MPa. Determine the change in diameter, change in length and change in volume af the shell. E=205 GPa and Poison's ratio 0.3.
5. Wall thickness of a cylindrical shell of 800 mm internal diameter and 2 m long is 10 mm. If the shell is subjected to an internal pressure of 1.5 Mpa. Find the following
   (i) The maximum intensity of shear stress induced
   (ii) The change in dimensions of the shell. Take E=200 Gpa and ν=0.3
6. A thin cylindrical tube with closed ends is subjected to an internal pressure of 8 Mpa. The tube is of 90 mm internal diameter and 6 mm thickness. Determine the maximum and minimum principal stresses and maximum shear stress if a torque of 4000 N-m is also applied on the tube.
7. Determine the thickness of metal required for a cylindrical shell of shell of steel of 180 mm internal diameter to withstand an internal pressure of 30 Mpa. The circumferential stress in the section must not exceed 150 Mpa.
8. A steel tube of 50 mm outer diameter is shrunk on another tube of 30 mm inner diameter and 40 mm outer diameter. The compound tube is made to withstand an internal pressure of 50 Mpa. The Shrinkage allowance is such that the final maximum stress in each tube is the same. Find this value of stress and show on diagram the variation of circumferential stress in the tube. Also find the initial difference of diameters before shrinkage on take E=208 Gpa.
9. The maximum principal strain in a thin cylindrical tank, having a radius of 25 cm and wall thickness of 5 mm when subjected to an internal pressure of 1MPa, is (taking Young's modulus as 200 GPa and Poisson's ratio as 0.2)
10. Circumferential stress in a cylindrical steel boiler shell under internal pressure is 80 MPa. Young's modulus of elasticity and Poisson's ratio are respectively 2 × 105 MPa and 0.28. The magnitude of circumferential strain in the boiler shell will be?
11. The percentage change in volume of a thin cylinder under internal pressure having hoop stress = 200 MPa, E = 200 GPa and Poisson's ratio = 0·25 is?
12. A thin cylinder of inner radius 500 mm and thickness 10 mm is subjected to an internal pressure of 5 MPa. The average circumferential (hoop) stress in MPa is? A thin cylinder contains fluid at a pressure of 500 N/m2, the internal diameter of the shell is 0.6 m and the tensile stress in the material is to be limited to 9000 N/m2. The shell must have a minimum wall thickness of nearly.
13. A thin cylinder of 100 mm internal diameter and 5 mm thickness is subjected to an internal pressure of 10 MPa and a torque of 2000 Nm. Calculate the magnitudes of the principal stresses
14. A thin cylindrical pressure vessel of inside radius „r" and thickness of metal „t" is subject to an internal fluid pressure p. What are the values of
    (i) Maximum normal stress?
    (ii) Maximum shear stress?
15. A cylindrical shell has the following dimensions:
    Length = 3 m
    Inside diameter = 1 m
    Thickness of metal = 10 mm
    Internal pressure = 1.5 MPa
    Calculate the change in dimensions of the shell and the maximum intensity of shear stress induced. Take E = 200 GPa and Poisson"s ratio **ν = 0.3**

B. Questions testing the analyzing / evaluating ability of students

1. A compound cylinder is made by shrinkage a cylinder of external diameter 300 mm and internal diameter 250 mm over another cylinder of external diameter and internal diameter 200 mm. the radial pressure at the junction after junction after shrinkage is 8 N/mm2. Find the final stress set up across the section, when the compound cylinder is subjected to an internal fluid pressure of 84.5 N/mm^2
2. A spherical shell of 1.2 m internal diameter and 6 mm thickness is filled with water under pressure until the volume is increased by $400*10^3$ mm^3. Find the pressure exerted by water on the shell. Take E=204 Gpa and Poisson's ratio is 0.3.
3. Determine the ratio of thickness to inner diameter of a tube subjected to internal pressure if the ratio of the internal pressure to the maximum circumferential stress is 0.5. For such a tube of 250 mm inside diameter, find the alterations of thickness of metal when the internal pressure is 80 Mpa, E=200 Gpa .
4. A cylinder of inner diameter 100 mm and external diameter of 200 mm subjected to internal fluid pressure of 10 N/mm2 and external fluid pressure of 20 N/mm2 respectively, calculate radial and hoop stresses developed in a cylinder wall at mean radius and also draw the variation of stresses across the wall.